Mathematical Tables
and other
Aids to Computation

A Quarterly Journal

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On Modified Divided Differences I

Although divided differences of a function are of basic importance in the theory of numerical analysis, they are not nearly as useful as ordinary differences in application. In the first place, they are much harder to generate; not only because they involve unequal intervals of the argument, but because divisions are involved, often by small numbers. Furthermore, they are harder to interpret than ordinary differences, and the problem of making judgments at each step about the number of meaningful significant figures in the successive differences is one of considerable difficulty for high-speed computing machines, or for I.B.M. machines. We shall show that part of this difficulty at least can be overcome by modifying the definition of divided differences. These *modified* divided differences take on many of the characteristics of ordinary differences, based on equal intervals between successive arguments, and the familiar theory relating to ordinary differences becomes a special case of the more general theory. Moreover, the decimal point can remain fixed, and all differences can be carried to the same number of *decimals* as the entries themselves. Such modified differences can therefore serve as useful tools in analyzing functions which are available only at unequal intervals of the argument.

Definitions. Let a function $u(x)$ be given for a sequence of arguments $x_0, x_1, \cdots, x_n$, all included in a closed region X. No assumptions need be made about the ordering in the sequence. For brevity, let $u_k = u(x_k)$. The accepted definition[1] for the first divided difference of u_k and u_j is

$$[x_k x_j] = \frac{u_k - u_j}{x_k - x_j}.$$

The divided difference of order m, involving $u_0, u_1, u_2, \cdots, u_m$ is obtained from the divided differences of order $m - 1$ by the formula

$$[x_0 x_1 x_2 \cdots x_m] = \frac{[x_0 x_1 \cdots x_{m-1}] - [x_1 x_2 \cdots x_m]}{x_0 - x_m}.$$

It is well known that $[x_0 x_1 \cdots x_m]$ is a symmetric function of the arguments. If $u(x)$ possesses n continuous derivatives in X, then

$$(1.1) \qquad [x_0 x_1 \cdots x_n] = \frac{1}{n!} \frac{d^n u(t)}{dt^n}, \quad t \text{ in } X.$$

In particular, if $u(x)$ is a polynomial of degree n, then the nth divided difference is a constant, given by the right-hand side of (1.1). In the following discussion it will always be assumed that the function under consideration has n continuous derivatives in X, whenever the nth difference (ordinary or divided) is referred to.

It is to be noted from (1.1) that the order of magnitude[2] of a divided difference does not depend strongly on the size of the intervals between

successive arguments. In contrast to this, the nth *ordinary* difference $\Delta^n u_0$ is related to the nth derivative of the function as follows:

$$\Delta^n u_0 = h^n \frac{d^n u(z)}{dz^n}, \quad z \text{ in } X, \tag{1.2}$$

where h is the constant difference between successive arguments. Hence $|\Delta^n u_0|$ *decreases* with h; this is a desirable property. In practice, h can often be chosen small enough so that an approximating polynomial of a given degree shall represent $u(x)$ with desired accuracy. We can introduce a *modified* divided difference possessing similar characteristics as follows:

Define the modified first divided difference of u_0 and u_1 by

$$[w; x_0x_1] = w[x_0x_1] = \frac{w}{x_1 - x_0}(u_1 - u_0), \tag{1.3}$$

where w is an arbitrarily chosen positive constant. We shall show in Section II that if there are N arguments x_k in X, then a useful constant to choose is one which is close in magnitude to

$$w = \sum_{k=1}^{N-1} \frac{|x_k - x_{k-1}|}{N - 1}. \tag{1.4}$$

If the arguments x_k form an increasing or decreasing sequence in X, then (1.4) becomes

$$w = |x_{N-1} - x_0|/(N - 1). \tag{1.5}$$

In many cases the order of magnitude of $x_k - x_{k-1}$ remains the same over the region X, even though the intervals are not all equal. If w is chosen as the average length of the grid intervals in X, $w/(x_1 - x_0)$ will usually be of the order of magnitude of unity. In that case the number of decimals in the first modified divided difference which have meaning is the same as in the entries $u(x)$ themselves.

We define

$$[w; x_0x_1 \cdots x_n] = \frac{nw}{x_0 - x_n}\{[w; x_0x_1 \cdots x_{n-1}] - [w; x_1x_2 \cdots x_n]\}. \tag{1.6}$$

Again $nw/(x_n - x_0)$ will usually be of the order of magnitude of unity, and it is reasonable to retain the same number of decimals in all the divided differences as there are in $u(x)$.

Since w was introduced as a device to permit the retention of a fixed decimal point, it is enough to use a value of w which is close to, but not necessarily equal to, that defined in (1.4).

It follows from (1.1) and (1.6) that

$$[w; x_0x_1 \cdots x_n] = \frac{w^n d^n u(t)}{dt^n}, \quad t \text{ in } X. \tag{1.7}$$

The character of the modified divided differences parallels that of ordinary differences, and the rate at which the successive modified divided differences fall off again permits an estimation of the accuracy of a given approximation formula for $u(x)$ over X. Let

$$x - x_0 = pw; \quad x_k - x_0 = p_k w.$$

The well known identity for divided differences now assumes the following form:

$$
\begin{aligned}
u(x) &= u_0 + p[w; x_0x_1] + \frac{p(p-p_1)}{2!}[w; x_0x_1x_2] \\
&\quad + \frac{p(p-p_1)(p-p_2)}{3!}[w; x_0x_1x_2x_3] + \cdots \\
&\quad + \frac{p(p-p_1)(p-p_2)\cdots(p-p_{n-1})}{n!}[w; x_0x_1x_2\cdots x_n] \\
&\quad + \frac{p(p-p_1)(p-p_2)\cdots(p-p_n)}{(n+1)!}[w; xx_0x_1\cdots x_n].
\end{aligned}
\tag{1.8}
$$

If the right-hand side of (1.8) is truncated after the divided difference of order n, then the remainder R can be expressed by

$$R = M[w; xx_0x_1\cdots x_n] = Mw^{n+1}\frac{d^{n+1}u(t)}{dt^{n+1}}, \quad t \text{ in } X, \tag{1.9}$$

where

$$M = \frac{p(p-p_1)(p-p_2)\cdots(p-p_n)}{(n+1)!}.$$

Whenever M is numerically less than unity, the magnitude of the divided differences of order $(n+1)$ provides an estimate for an upper bound of the truncation error R.

We can write

$$[w; x_0x_1x_2\cdots x_m] = \sum_{k=0}^{m}\frac{w^m m!\,u_k}{\prod_{j=0}^{m}{}'_k(x_k - x_j)}, \tag{1.10}$$

where the prime indicates that the factor corresponding to $j = k$ is to be omitted. In the special case of a divided difference of *even* order $2n$, chosen centrally around x_0, (1.10) becomes

$$[w; x_{-n}x_{-n+1}\cdots x_0x_1\cdots x_n] = \sum_{k=-n}^{n}\frac{(-1)^{n-k}w^{2n}\binom{2n}{n+k}u_k}{\prod_{j=-n}^{n}{}'_k\left(\frac{x_k - x_j}{k-j}\right)}, \tag{1.11}$$

$$[w; x_{-n}x_{-n+1}\cdots x_0\cdots x_n] = \sum_{k=-n}^{n}\frac{\binom{2n}{n+k}(-1)^{n-k}u_k}{\prod_{j=-n}^{n}{}'_k\left(\frac{p_k - p_j}{k-j}\right)}. \tag{1.12}$$

In the case where all intervals are equal to w, $p_k = k$, and (1.12) reduces to the ordinary central difference.

Since the only difference between the modified and ordinary divided difference of order m is the factor $w^m m!$ for all entries in a column of differences, it would appear that just as much information could be obtained from the entries without modification, provided the proper number of significant figures were retained in each difference. This is true. The chief advantages of modified divided differences over ordinary divided differences can be summarized as follows:

1) If w is suitably chosen the modified differences can be retained to the same number of places as the entries u_n themselves.

2) The numbers $nw/(x_k - x_j)$ by which one multiplies are generally of the order of magnitude of unity; hence it is easy to program their computation for high speed machines.

3) In computing $u(x)$ from (1.8) the magnitudes of the first neglected divided differences in the region furnish an upper bound of the error, if $|M|$ in (1.9) is no greater than unity (a case of frequent occurrence).

4) The character of the differences is close to that of the familiar ordinary differences based on equal intervals between successive arguments.

In employing modified divided differences over a large region X, it is often desirable to consider several subregions, in each of which w is chosen to be close to the average length of the interval over the subregion.

Error Magnification. Consider the ordinary central difference of *even* order $2n$, based on the entries $u_{-n}, u_{-n+1}, \cdots, u_n$—usually denoted by $\delta^{2n}u_0$. From (1.12) it is seen that the entry u_0 appears in this difference with the coefficient $(-1)^n\binom{2n}{n}$. In the neighboring differences $\delta^{2n}u_k$, $|k| < n$, u_0 appears with the coefficient $(-1)^{n-k}\binom{2n}{n+k}$. It follows that if u_0 has an error ϵ_0 which is substantially larger numerically than $|\delta^{2n}u_k|/\binom{2n}{n}$, and if the errors in neighboring entries are of a lower order of magnitude, then the differences $\delta^{2n}u_k$ will alternate[3] in sign, and their numerical ratios to $\delta^{2n}u_0$ will be approximately $\binom{2n}{n-k}/\binom{2n}{n}$. Since the error ϵ_0 is magnified most in $\delta^{2n}u_0$, an incorrect entry can be picked out immediately, and even the order of magnitude of the error can be estimated from the difference pattern. (A divergence of the pattern from that of the binomial coefficient ratios indicates the presence of more than one error.) If a regular function $u(x)$ is tabulated at equally spaced arguments, systematic differencing of the entries constitutes the most powerful tool for checking the function.

The corresponding modified divided differences, based on unequal intervals between successive arguments, are inherently more difficult to interpret from the viewpoint of error indication. However, certain useful inequalities can be set down. In order to derive them, we shall limit the region X, and specify that the arguments x_k form either an increasing or a decreasing sequence. In application this is actually the most important case; and in a region where $x_k - x_{k-1}$ changes sign for some k, we may consider the subregions in which the sign of $x_k - x_{k-1}$ is constant. Furthermore, no essential restriction will be added if for convenience we consider only the case where $x_k > x_{k-1}$; this will now be assumed.

Just as in the case of ordinary differences, u_k enters into $2n + 1$ consecutive differences of order $2n$. It will be enough to examine a fixed u_0 and

to study the coefficients with which it enters into the various differences of order $2n$. Let us write

$$\delta_w^{2n} u_0 = [w; x_n x_{n-1} \cdots x_0 \cdots x_{-n}] = \sum_{k=-n}^{n} M_{0,k}^{(2n)} u_k, \tag{2.1}$$

and more generally

$$\delta_w^{2n} u_m = [w; x_{m-n} x_{m-n+1} \cdots x_{m+n}] = \sum_{k=-n}^{n} M_{m,k+m}^{(2n)} u_{m+k}. \tag{2.2}$$

When no ambiguity is likely to arise, the superscripts will be dropped and we shall write $M_{m,k}$ for $M_{m,k}^{(2n)}$.

Let

$$w c_k = x_k - x_{k-1}.$$

Clearly c_k is positive, since $x_k - x_{k-1}$ is assumed positive.

From (1.11) we have

$$M_{0,0} = (-1)^n \binom{2n}{n} / V_{0,0}, \tag{2.3}$$

where

$$V_{0,0} = c_0 c_1 \left(\frac{c_0 + c_{-1}}{2}\right)\left(\frac{c_1 + c_2}{2}\right)\left(\frac{c_0 + c_{-1} + c_{-2}}{3}\right)\left(\frac{c_1 + c_2 + c_3}{3}\right) \cdots$$
$$\times \left(\frac{c_0 + c_{-1} + \cdots + c_{-n+1}}{n}\right)\left(\frac{c_1 + \cdots + c_n}{n}\right),$$

$$\frac{M_{k,0}}{M_{0,0}} = \frac{(-1)^k \prod_{j=-nc}^{(-n-1+ck)c} (x_0 - x_j)}{\prod_{j=(n+1)c}^{(n+kc)c} (x_j - x_0)}, \begin{cases} 1 \leqq k^2 \leqq n^2 \\ c = 1, \text{ if } k \geqq 1. \\ c = -1, \text{ if } k < 0 \end{cases} \tag{2.4}$$

In particular

$$\frac{M_{1,0}}{M_{0,0}} = \frac{-(x_0 - x_{-n})}{(x_{n+1} - x_0)}\,; \quad \frac{M_{-1,0}}{M_{0,0}} = \frac{-(x_n - x_0)}{(x_0 - x_{-n-1})}. \tag{2.5}$$

From (2.4) we have

THEOREM 1. *The coefficients $M_{k,0}$ of u_0 alternate in sign.*

THEOREM 2. *If $p \leq c_k \leq P$, for $-n \leq k \leq n$, then*

$$\binom{2n}{n}\frac{1}{P^{2n}} \leq (-1)^n M_{0,0} \leq \binom{2n}{n}\frac{1}{p^{2n}}, \tag{2.6}$$

$$\frac{n}{n+1}\frac{p}{P} \leq \frac{-M_{\pm 1,0}}{M_{0,0}} \leq \frac{n}{n+1}\frac{P}{p}, \tag{2.7}$$

$$\frac{\binom{2n}{n+k}}{\binom{2n}{n}}\left(\frac{p}{P}\right)^k \leq \frac{(-1)^k M_{\pm k,0}}{M_{0,0}} \leq \frac{\binom{2n}{n+k}}{\binom{2n}{n}}\left(\frac{P}{p}\right)^k. \tag{2.8}$$

The above inequalities follow from (2.3), (2.4), and (2.5).

If $P = p = 1$ the inequalities give the well known relations among the coefficients of the ordinary central differences. From (2.6) and (2.8) we have

$$\frac{p^k}{P^{2n+k}}\binom{2n}{n+k} \leq |M_{\pm k, 0}| \leq \binom{2n}{n+k}\frac{P^k}{p^{2n+k}}. \tag{2.9}$$

THEOREM 3. *If* $P \leqq \{2(2m+1)/(m+1)\}^{\frac{1}{2}}$, *the coefficients* $|M_{0,0}^{2n}|$ *increase with* n, *for* $n \geqq m$.

Proof.

$$|M_{0,0}^{(2n+2)}| = \frac{|M_{0,0}^{2n}|\,w^2(2n+2)(2n+1)}{(x_0 - x_{-n-1})(x_{n+1} - x_0)} \geq \frac{|M_{0,0}^{2n}|\,2(2n+1)}{P^2(n+1)}$$

$$\geq |M_{0,0}^{(2n)}|\frac{2n+1}{2m+1}\,\frac{m+1}{n+1} \geq |M_{0,0}^{(2n)}|,$$

since $2n + 1/(n+1)$ is an increasing function of n.

COROLLARY 3.1. *If* $p \geq \{2(2n+1)/(n+1)\}^{\frac{1}{2}}$, $|M_{0,0}^{(2n)}|$ *decreases with* n.

Before applying the previous results, it will be desirable to distinguish between three types of errors in the entries, just as in the case when ordinary differences are under consideration:

Errors of Type (a). Systematic errors, prevalent in all entries over a region. Such errors cannot be discovered by differencing methods; hence they will be eliminated from the discussion.

Errors of Type (b). Under this category we shall include

1. An isolated error: An error in an entry—say in u_0—which is considerably larger numerically than the errors of most entries in the region.
2. Multiple errors: This term will be used when two or more entries in a region have errors which are considerably larger numerically than the errors of most entries in the region.

The phrase *considerably larger* is necessarily qualitative, and depends on what degree of magnitudes the tests applied can distinguish.

Errors of Type (c).

1. Errors due to rounding approximate entries to a fixed number of decimals.
2. Errors too small to be considered in Type (b).
3. Errors of a non-systematic type which occur in too many entries of a region to be picked up by differencing tests. Usually all errors of Type (c) will occur in the last two decimal places of the entries. A discussion of errors of Type (c) will be given in Part II of this paper.

Errors of Type (b). Consider an isolated error ϵ_0 in u_0. Let $\bar{u}_0 = u_0 + \epsilon_0$ be the tabulated entry, and suppose that $M_{0,0}^{(2n)}\epsilon_0$ is of a higher order of magnitude numerically than $\delta_w^{2n} u_k$.

Let $\bar{\delta}_w^{2n} u_k$ be the approximate value of the $(2n)th$ divided difference in the region, and let

$$S\varphi_0^{(2n)} = \delta_w^{2n} u_0 - \delta_w^{2n} u_k, \quad S = \text{signum}\,[\delta_w^{2n} u_0 - \bar{\delta}_w^{2n} u_k]. \tag{2.10}$$

On the assumption that $S\varphi_0^{2n}$ is due primarily to the error, we may write

$$(-1)^n \rho\epsilon_0 M_{0,0}^{(2n)} = S\varphi_0^{(2n)},$$

where ρ is a positive number close to unity.

(2.11) $$\left|\frac{\varphi_0^{(2n)}}{M_{0,0}}\right| = |\rho\epsilon_0|.$$

From (2.6)

(2.12) $$\frac{\varphi_0^{(2n)}p^{2n}}{\binom{2n}{n}} \leq |\rho\epsilon_0| \leq \frac{\varphi_0^{(2n)}P^{2n}}{\binom{2n}{n}}.$$

The estimation of $\delta_w^{2n}u_k$ for the region can be avoided. For we may study

(2.13) $$\delta_w^{2n}u_0 - \delta_w^{2n}u_1 = \rho S\varphi_{0,1}^{(2n)}, \text{ say.}$$

If ϵ_0 is an isolated error, then

(2.13a) $$(M_{0,0} - M_{1,0})\epsilon_0\rho \cong S\varphi_{0,1}^{(2n)},$$

$$\rho S(-1)^n\epsilon_0 \cong \frac{\varphi_{0,1}^{(2n)}}{(-1)^n M_{0,0}\left(1 - \dfrac{M_{1,0}}{M_{0,0}}\right)}.$$

From (2.7)

$$\frac{(-1)^n\varphi_{0,1}^{(2n)}}{M_{0,0}\left(1 + \dfrac{n}{n+1}\dfrac{P}{p}\right)} \leq \rho S(-1)^n\epsilon_0 \leq \frac{(-1)^n\varphi_{0,1}^{(2n)}}{M_{0,0}\left(1 + \dfrac{n}{n+1}\dfrac{p}{P}\right)},$$

(2.14) $$\frac{\varphi_{0,1}^{(2n)}p^{2n}}{\binom{2n}{n}\left(1 + \dfrac{n}{n+1}\dfrac{P}{p}\right)} \leq \rho S(-1)^n\epsilon_0 \leq \frac{\varphi_{0,1}^{(2n)}P^{2n}}{\binom{2n}{n}\left(1 + \dfrac{n}{n+1}\dfrac{p}{P}\right)}.$$

THEOREM 4. *If ϵ_0 is an isolated error, then*

(2.15) $$\frac{\delta_w^{2n}u_0 - \delta_w^{2n}u_1}{\delta_w^{2n}u_0 - \delta_w^{2n}u_{-1}} \cong \frac{1 + \dfrac{x_0 - x_{-n}}{x_{n+1} - x_0}}{1 + \dfrac{x_n - x_0}{x_0 - x_{-n-1}}}.$$

Proof.

$$\delta_w^{2n}u_0 - \delta_w^{2n}u_1 \cong \epsilon_0(M_{0,0} - M_{1,0}) \cong \epsilon_0 M_{0,0}\left(1 - \frac{M_{1,0}}{M_{0,0}}\right),$$

$$\delta_w^{2n}u_0 - \delta_w^{2n}u_{-1} \cong \epsilon_0 M_{0,0}\left(1 - \frac{M_{-1,0}}{M_{0,0}}\right).$$

Then (2.15) follows from (2.4) and the above. Since the right-hand side (2.15) is easy to evaluate, we have a criterion as to whether a possible error is an isolated one. The estimate of the error is furnished by (2.12) or (2.14).

Let us now examine the effect of w on the location of errors. From the right-hand side of (2.10) it is clear that if w is chosen excessively small, then significant figures of u_k will be moved off quickly to the right and it will be impossible to pick up significant errors of the last few places of the entries. Moreover, both P and p will tend to be large, since $x_k - x_{k-1} = c_k w$

is fixed by the given grid for all k. In that case (2.14) will furnish a poor estimate of ϵ_0, due to the sharp variations in the inequalities. On the other hand, the choice of a very large w will simply have the effect of magnifying both small and large errors. A desirable w is one which tests as well as possible all the significant figures of the entries u_k. For that purpose, w should be chosen as the *average* $|x_k - x_{k-1}|$ in the region. For then both P and p will tend to be of the order of magnitude of unity, provided successive intervals vary with reasonable continuity.

If w is chosen as the average of the values $x_k - x_{k-1}$, then there may be some regions where P is relatively large with respect to w; and other regions where P is quite small. The question arises: If there is an error of Type (b), will (2.13) show it? From (2.13a) and (2.3) it is clear that $\varphi_{1,0}^{(2n)}$ will tend to be small in regions where the intervals $x_k - x_{k-1}$ are *large* with respect to the average interval of the space. Hence a small error will not show up. If the spread between P^{2n} and p^{2n} is small, in such a region then (2.14) can be used, to examine critically a fairly small $\varphi_{0,1}^{(2n)}$. However, if an isolated group of $k < n$ intervals is excessively large, then p and P will vary considerably, and the work of testing every doubtful entry may become too laborious. By the same considerations, a region where the values of $x_k - x_{k-1}$ are excessively small with respect to the average will show a big error magnification, and such errors will be readily traceable. Example 2 illustrates some features of error patterns.

If P varies much from p, the inequalities given here will not offer a good estimate of the magnitude of an isolated error. However, from Theorem 1 it is assured that a sufficiently large error will show up, because of sign variation in the vicinity of an error, if the magnitude of the modified divided difference is of lower order than the error. Small errors, on the other hand, can be masked by the rounding errors in the vicinity, often to a greater degree than in the case of equally spaced entries.

Let us now examine briefly the consequences of Theorem 3. In ordinary differences, where all c_k are unity, the numerical value of $M_{0,0}$ increases with n. Thus the magnification of an error increases in successive differences. If differences of high enough order are taken and the interval is sufficiently small for the correct differences to fall off numerically, an isolated error will be noticed more and more in successive differences. To have this property carry over to regions where all c_k are not equal to unity, Theorem 3 requires that no c_k be numerically greater than $2[(n + \frac{1}{2})/(n + 1)]^{\frac{1}{2}}$. Thus in regions where the intervals between successive entries are numerically larger than twice the *average* interval in the region, successive differencing will not tend to magnify the error, and so detection of the error may become difficult. It would also be desirable to have the ratio $|M_{\pm 1,0}/M_{0,0}|$ as small as possible. If the c_k are all equal to unity, this ratio is $n/(n + 1)$ for the difference of order $2n$. Hence with increasing n, this ratio is so close to unity as not to offer much help in the *variation of magnification* even if all c_k are unity. However, the fact that the *sign of* $M_{\pm 1,0}/M_{0,0}$ *is always negative* is of the utmost importance. For even if this ratio is numerically close to unity, the fact that successive $M_{k,0}$ differ in sign will help in detecting an error—especially so in regions where $|M_{0,0}|$ increases with n.

Although attention was fixed on even central differences, the main characteristics apply also to differences of odd order.

The following examples are instructive.

Example 1

(a) Unmodified Divided Differences of $Y_0(x)$

x	$Y_0(x)$						
1.00	.08825696						
		.7640470					
1.04	.11881884		−.42250				
		.742922		.124			
1.05	.12624806		−.41258		−.10		
		.7264190		.1038		.1	
1.08	.14804063		−.395967		−.0800		—
		.6670239		.086985		.0600	
1.20	.22808350		−.378570		−.06561		−.021
		.6026670		.071895		.0540	
1.25	.25821685		−.364179		−.05102		−.047
		.5735327		.05965		.0390	
1.28	.27542283		−.357021		−.0397		−.02
		.5485412		.05290		.034	
1.32	.29736448		−.350673		−.0326		—
		.5169806		.04769		.03	
1.37	.32321351		−.344473		−.0266		
		.4859780		.04343			
1.41	.34252663		−.339261				
		.4622297					
1.44	.35651952						

Comment. In obtaining the unmodified divided differences, the computer was instructed to retain in $[x_ix_{i+1} \cdots x_{i+n+1}]$ the same number of significant figures as there were in $[x_ix_{i+1} \cdots x_n] - [x_{i+1} \cdots x_{n+1}]$. Another reasonable method is to retain in a column only the number of significant figures that are common to most entries in the column. The difficulty here is that the computer does not know, until the column is finished, what this common number should be.

(b) Modified Divided Differences of $Y_0(x)$, in Units of the 8th Decimal Place

$w = 0.044$

x	$Y_0(x)$	δ_w	δ_w^2	δ_w^3	δ_w^4	δ_w^5	δ_w^6
1.00	.08825696						
		3361807					
1.04	.11881884		−163592				
		3268857		6338			
1.05	.12624806		−159751		−900		
		3196244		5305		198	
1.08	.14804063		−153318		−720		−74
		2934905		4446		119	
1.20	.22808350		−146582		−590		−11
		2651735		3675		107	
1.25	.25821685		−141010		−459		−24
		2523544		3049		077	
1.28	.27542283		−138239		−357		− 8
		2413581		2704		067	
1.32	.29736448		−135781		−293		− 9
		2274715		2437		059	
1.37	.32321351		−133380		−239		
		2138303		2219			
1.41	.34265263		−131362				
		2033811					
1.44	.35651952						

Example 2

Modified Divided Differences of x^3, in Units of the Fifth Decimal Place

$w = 0.35$

x	x^3	δ_w	δ_w^2	δ_w^3	δ_w^4
−2.4	−13.82400				
		509600			
−2.0	− 8.00000		−139106		
		330750			
−1.5	− 3.27500		−124950		
		170100			
−1.1	− 1.33100		− 71595		
		78050		11434	
− .6	− .21600		− 44100		22231
		15050		40016	
.1	− .00100		− 12250		−14291
		1050		20621	
.2	.00800		14700		4203
		13650		25725	
.5	.12500		93100		0
		66850		25725	
.6	.31600		−200900		
		9450			
.7	.34300		117600		
		76650			
1.0	1.00000		75950		
		152600			
1.4	2.74400				

Comments on Example 2. The entries corresponding to $x = -1.5$ and 0.6 contain errors of the same magnitude. Yet the error in $u(.6)$ is already evident from $\delta_w^2 u$, but not so the error in $u(-1.5)$. This illustrates the fact that errors are magnified most in regions where the intervals $c_k w$ are small with respect to w. Let

$$x = .6 = x_0, \quad n = 1; \quad \text{then} \quad \frac{x_0 - x_{-1}}{x_2 - x_0} = 1/4 = \frac{x_1 - x_0}{x_0 - x_{-2}}$$

using (2.15),

$$\left(1 + \frac{x_0 - x_{-n}}{x_{n+1} - x_0}\right) \Big/ \left(1 + \frac{x_n - x_0}{x_0 - x_{-n-1}}\right) = 1.$$

From the table of differences, the left-hand side of (2.15) for this example is equal to

$$\frac{-3.185}{-2.940} = 1.08$$

Hence the error appears to be an isolated one.

Let us now consider what information can be obtained from (2.12) about the nature of the error. In the usual case, the true value of $\delta_w^{(2n)}$ is unknown, and $\varphi_0^{(2n)}$ is estimated from adjacent differences. In the present case, the second differences at $x = 0.1$ and 0.2 warrant a guess that the second modified difference at $x = 0.6$ is about 0.4. This is based on the assumption that the correct third modified difference in the region is about 0.25. Since, from (1.6),

$$[w; x_0x_1x_2x_3] \frac{(x_3 - x_0)}{3w} + [w; x_0x_1x_2] = [w; x_1x_2x_3]$$

and $(x_3 - x_0)/3w$ is approximately $\frac{1}{2}$ in this region, we have, applying the above formula twice,

$$\delta_w^2 u \cong 0.15 + 0.25 = 0.4, \quad \text{at } x = 0.6.$$

Using the above estimate in (2.12) we arrive at the following calculations:

$$P = .86; \quad p = .285; \quad P^2 = 0.74; \quad p^2 = 0.081,$$
$$|\varphi^{(2)}| \cong |-2.009 - 0.4| = 2.409; \quad |\varphi^{(2n)}/\tbinom{2n}{n}| \cong 1.205.$$

Thus (2.12) implies

$$1.205(.081) \leq |\epsilon| \leq 1.205(.74); \quad \text{or} \quad 0.097 \leq |\epsilon| \leq 0.90.$$

The true error does lie between the above limits, but the actual size of the error must be obtained from recomputation. The important fact to observe is that the pattern of the second differences correctly shows that $u(0.6)$ is in error.

The error in $u(-1.5)$ shows up in the fourth difference.

Example 3. The following four values of $f(y)$ are available.

y	$f(y)$	Δ	Δ^2
3.7416573868	+.00824 2550		
3.777	+.00003 5971	−230753	+79
3.778	−.00019 4782	−230674	
3.779	−.00042 5456		

The differences of the equally-spaced values have been entered. It is required to find the zero of $f(y)$. If it were certain that the three last values of $f(y)$ are correct, and that $\Delta^3 y$ is negligible, this zero could be easily obtained by quadratic inverse interpolation. However, it happens that the computation of $f(y)$ is quite laborious, and it is desirable to check the accuracy of $f(y)$. Let us obtain modified divided differences, using the interval $w = .001$, since in this problem we merely want to have an indication of the magnitude of $\Delta^3 f$ in this region, at an interval of .001 in y. The values follow.

$w = .001$

y	$f(y)$	δ_w	δ_w^2
3.7416573868	+.00824 2550		
		−232201	
3.777	+.00003 5971		80
		−230753	
3.778	−.00019 4782		79
		−230674	
3.779	−.00042 5456		

The fact that the two values of δ_w^2 differ by only one unit is assurance that the values of $f(y)$ are correct, and that quadratic inverse interpolation is adequate. The solution is $y_0 = 3.77715\ 586$, to eight decimal places. [It can be shown that $\lambda = \frac{1}{4}y_0^2$ satisfies the system $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda\right) u = 0$, $u = 0$ on the boundary C, $u > 0$ in interior of region where C is the ellipse $\frac{1}{4}x^2 + y^2 = 1$.]

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[1] L. M. MILNE-THOMSON, *The Calculus of Finite Differences*. London and New York, 1933.

[2] The numbers u_1 and u_2 will be said to have the same order of magnitude if $1/10 < |u_1/u_2| < 10$.

[3] See J. C. P. MILLER, "Checking by Differences I," *MTAC*, v. 4, 1950, p. 3–11.

A Minimum Problem Solved by Mesh Methods

Introduction. In the following, the function $y(x)$, subject to $y(0) = y(1) = 1$, is sought which will minimize the integral

$$I = \int_0^1 y^{-1}(1 + y'^2)^{\frac{1}{2}}\, dx. \tag{1}$$

This problem can be solved by the usual methods of the calculus of variations[1] but the differential equation involved is rather complicated. It is proposed here to solve the problem by mesh or "assumed polynomial" methods.[2] Methods of this sort have been rather extensively applied to the solution of differential equations and boundary value problems, and, recently, also to the determination of characteristic numbers or eigenvalues.[3]

Basic Equations. To solve the above problem by mesh methods it is first assumed that the interval $0 \leq x \leq 1$ is divided into segments each of equal length $h = 1/(2n)$ by the points x_i, $i = 0(1)2n$. It should be noted

that the solution will be symmetric about $x = \frac{1}{2}$; hence only the portion $0 \leq x \leq \frac{1}{2}$ need be considered together with the condition

$$y'(\tfrac{1}{2}) = 0. \tag{2}$$

We then write, using approximate integration,

$$I \approx 2 \sum_{i=1}^{n} 2hr_i/(y_i + y_{i-1}) \tag{3}$$

where

$$r_i = \{1 + h^{-2}(y_i - y_{i-1})^2\}^{\frac{1}{2}} \tag{4}$$

so that I is now a function of y_j for $j = 0(1)n$. At a minimum we must have

$$\frac{\partial I}{\partial y_k} = 0, \quad k = 1, 2, \cdots, n-1. \tag{5}$$

If we write

$$s_k = y_k + y_{k-1}, \qquad g_k = (r_k/s_k) + (r_{k-1}/s_{k-1}),$$
$$\rho_k = s_{k+1}r_{k+1}/(s_k r_k), \qquad e_k = -h^2 r_k s_k g_k,$$

we can write equation (5) as

$$y_{k+1} = y_k + \rho_k(y_k - y_{k-1}) + e_{k+1}. \tag{6}$$

The process of solution followed here is to assume a curve $y(x)$ and use this to determine the ρ's and e's at mesh points. Then (6) provides a recurrence relation for determining new values of $y_2, y_3, \cdots, y_n$ which we write as $Y_2, Y_3, \cdots, Y_n$ since the mid-point slope condition (2) is not as yet satisfied. Using second order deriving coefficients as given by SOUTHWELL[2] or MILNE[4] (p. 96–98) we calculate Y_n' which usually is not zero. Then, letting $y_i' = Y_i' - Y_n'$ so as to satisfy $y_n' = 0$ we integrate y_i' by use of integrating coefficients[2,4] to get a new set of y's. The above process is then repeated.

The accompanying table shows results for three trials using $h = 1/8$ and starting with $y \equiv 1$. An approximate value of $\frac{1}{4}I$ was found for each assumed curve $y(x)$. These successive values, i.e. .5, .48148, .48144, indicate a minimum being reached.

	k	y_k	$y_k - y_{k-1}$	r_k	s_k	ρ_k	g_k	e_k	Y_k	Y_k'	y_k
	0	1.00000	—	—	—	—	—	—	1.00000	+.0625	1.00000
	1	1.00000	.00000	1.00000	2.00000	1.00000	—	—	1.00000	−.0625	1.05469
Run 1	2	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.984375	−.1875	1.09375
	3	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.953125	−.3125	1.11719
	4	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.906250	−.4375	1.12500
	0	1.00000	—	—	—	—	—	—	1.00000	.50504	1.00000
	1	1.05469	.05469	1.09152	2.05469	1.003638	—	—	1.05469	.37000	1.05362
Run 2	2	1.09375	.03906	1.04769	2.14844	.99937	.48553	−.01708	1.09250	.24124	1.09071
	3	1.11719	.02344	1.01743	2.21094	.99870	.43512	−.01529	1.11500	.12268	1.11241
	4	1.12500	.00781	1.00195	2.24219	—	.40744	−.01430	1.12347	.00804	1.11953
	0	1.00000	—	—	—	—	—	—	1.00000	.49656	1.00000
	1	1.05362	.05362	1.08812	2.05362	1.00096	—	—	1.05362	.36136	1.05358
Run 3	2	1.09071	.03709	1.04309	2.14433	.99971	.48486	−.01695	1.09034	.23280	1.09064
	3	1.11241	.02170	1.01496	2.20312	.99976	.43596	−.01523	1.11182	.11448	1.11234
	4	1.11953	.00712	1.00162	2.23194	—	.41018	−.01433	1.11896	−.00024	1.11947

Conclusions. It can be seen from the table that the process converges with fair rapidity. Further calculations by the author indicate that the results obtained after three trials contain an error of less than .001 for $y(\frac{1}{2})$. If the usual methods of the calculus of variations were employed the resulting non-linear differential equation would presumably have to be solved by finite difference methods anyway and it does not appear that this would be as easy a computation to carry through as the above.

In writing (3), first order divided differences have been used, these being the simplest and at the same time adequate. Higher order expressions for the derivatives may be employed but will in general result in more complicated recurrence relations. The iterative procedure for the solution apparently must be devised anew for each different class of problems.

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[1] R. WEINSTOCK, *Calculus of Variations*. New York, 1952.
[2] R. V. SOUTHWELL, *Relaxation Methods in Theoretical Physics*. Oxford, 1946.
[3] L. COLLATZ, *Numerisches Behandlung von Differentialgleichungen*. Berlin, 1951.
[4] W. E. MILNE, *Numerical Calculus*. Princeton, 1949.

A Modification of the Aitken-Neville Linear Iterative Procedures for Polynomial Interpolation

A. C. AITKEN[1] has described a method of interpolation which is equivalent to the use of Lagrange's polynomial formula but consists principally of the repeated computation of an attractively simple algorithm, well suited to desk calculators. The method does not require uniform spacing in the values of the argument at the points at which the values of the required function are given, though uniformity permits of a convenient check of some aspects of the calculation. It can therefore be used for both direct and inverse interpolation, and it is particularly valuable for the latter. The procedure depends on the following property, which is also discussed at greater length by E. H. NEVILLE[2] and by W. E. MILNE:[3]

On the basis of the known values of a function u at n values of the argument X—that is, at a point P and $n - 1$ other points, denoted collectively by Q—we may have obtained a polynomial interpolate u_{PQ} of degree $n - 1$ for the value of the function at some other point $X = x$. We may have obtained also another interpolate u_{QR} of degree $n - 1$, for the same value x, on the basis of the $n - 1$ points Q and a further point R. Then the polynomial interpolate of degree n for $X = x$, based on all $n + 1$ points P, Q, R, is

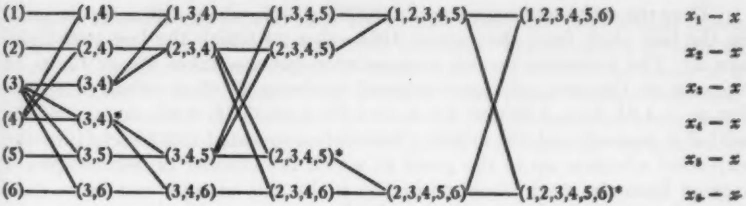
$$u_{PQR} = \begin{vmatrix} u_{PQ} & x_P - x \\ u_{QR} & x_R - x \end{vmatrix} \Big/ (x_R - x_P).$$

In this formula, x_P and x_R are the values of the argument at the points P and R respectively. It is easily computed on most desk calculators, espe-

cially as the divisor $x_R - x_P$ can be found (or checked), without extra work, as the net total of the two multipliers $x_R - x$ and $-(x_P - x)$ which are used in evaluating the determinant. Moreover, the linearity of the formula permits u_{PQ} and u_{QR} to be adjusted equally by any suitable amount before interpolation, thus minimizing the number of digits to be used at each stage. We use the term 'interpolation' loosely here to include extrapolation, which strictly occurs when $x_P - x$ and $x_R - x$ have the same sign. Aitken[1] calls this elementary process 'linear iteration by proportional parts,' and has also (according to WOMERSLEY[4]) called the derived value of u_{PQR} a 'linear cross-mean' of u_{PQ} and u_{QR}.

This iterative linear process or algorithm can therefore be used to increase the order of approximation of an interpolated value for a function, starting with a simple linear interpolation between two known values ($n = 1$) and progressively incorporating others. The final result, when the same known values have all been used, will necessarily be the same (apart from rounding-off variations) regardless of the order in which they are incorporated. The algorithm will be computed $\frac{1}{2}n(n - 1)$ times to incorporate n points, but with a suitable schedule considerable economies and simplifications can be effected. Aitken[1] proposes a schedule in which the work sheet for the computations can be laid out in such a way that the values required for each determinant always appear at the corners of rectangles. Neville[2] proposes a schedule which lacks this advantage but which usually gives a more rapidly convergent process. Neville's schedule permits of the addition of further known points, if desired, after the completion of the calculation of an interpolate, with greater freedom than Aitken's does, owing to the fundamental asymmetry of the latter. As Neville[2] and KINCAID[5] have demonstrated, Neville's method is also readily adapted to utilize the known values of differential coefficients at points at which the function is known.

The convenience of Aitken's schedule as regards the determinants, and much of the convenience of Neville's as regards the latitude in the addition of further points, are possessed by a third schedule which is also usually a little more rapidly convergent than Neville's in the early stages. With a condensed notation, using $(i, j, k, \cdots)$ to stand for the interpolate based on the points $X = x_i, x_j, x_k, \cdots$, the new procedure follows the schedule shown diagrammatically below for the case $n = 6$. The schedule for smaller values of n is the appropriate fragment of this one, and the procedure generalizes immediately to greater n. The cross products involving each line are taken with the items nearest the axis of symmetry of the schedule, and alternately on the further and nearer sides of this axis for successive columns, as is indicated by the lines in the diagram. In this diagram, the value of each interpolate $(i, j, k, \cdots)$ is written on the same line as one of the values on which it is directly based. Certain entries, marked by asterisks, are duplicated so that the values used in the determinant always occur at the corners of a rectangle. These entries can be computed separately as a partial check without interrupting the rhythm of the process. Incidentally, the geometrical interpretation given by Aitken[1] (p. 73–74) for his schedule can be adapted to represent the new schedule, and also to represent Neville's.



The new schedule can be regarded as a modification of Aitken's to introduce properties of symmetry which are often—though not always—desirable in an interpolation formula. Both for ease of computation and for accuracy of interpolation of a given order, it is normally most satisfactory when the numbers of given points on each side of the required point are either equal or differ by unity. That is to say, the required point should lie in the interval enclosed by the two given points which lie nearest to the axis of symmetry of the general pattern. Then the linear cross-mean procedure used at each stage is equivalent to an interpolation (for the columns of interpolates of odd degree) or an extrapolation (for the columns of interpolates of even degree) on a straight line fitted as a chord to a curve representing the previous column of values, and this concept is useful in judging the closeness of approximation in the new method. Moreover, the latter is designed to make the fullest use of the points nearest the required value from the earliest stages onwards, and to use them for interpolation rather than for extrapolation. Both of these features normally increase the rate of convergence, and thus facilitate both the computations and the detection of errors.

In the six-point example used by Milne[3] to illustrate Aitken's and Neville's methods, the range of variation of the results in the earlier columns is considerably reduced under the new schedule. We shall not, however, use this as an example here, since the given values of the function are at uniformly spaced values of the argument, and the NBS *Tables of Lagrangian Interpolation Coefficients*[6] enable the final interpolate to be found in two minutes easily with a calculator. The time taken to compute the fifteen linear cross-means under Aitken's or Neville's or the new schedule cannot practicably be reduced much below ten minutes (including writing down the necessary digits of the intermediate interpolates) with a fast electrical desk calculator.

Whittaker & Robinson[7] demonstrated a method of inverse interpolation for finding the positive root of the equation $u^7 + 28u^4 - 480 = 0$, given the values of $x = u^7 + 28u^4 - 480$ at the five points $u = 1.90(.01)1.94$. Aitken[1] (p. 71) and Neville[2] (p. 94–97) used this example to illustrate their schedules of linear iteration by proportional parts. Working to 10D for u, with 8D for x, the new schedule may be applied to solve this problem as follows:

u_i	$u = 1.9$	22	88	41	$x - x_i$
1.90	2271 69289	88 48558	41731	527	−25.71402 610
1.91	2279 04030	88 45462	41643		−14.62541 674
1.92	2286 32562				− 3.30746 392
1.93	2286 32562*				+ 8.24394 354
1.94	2283 41050	88 36413	41643*		+20.03298 301

Thus the estimated root is $u = 1.92288\ 41527$, which differs by six units in the last place from the correct 10D value, of which the last two digits are 33. The complete inverse interpolation process takes about 12 to 15 minutes on the desk calculator referred to above. Further values of x_i, say for $u_i = 1.95$ first, followed by $u_i = 1.89, 1.96, 1.88, 1.97$, etc., could be added if desired, and the existing results incorporated completely into the extended schedule up to the point at which the number of decimal places carried from the start ceases to justify it. In this example not more than two values can advantageously be added.

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[1] A. C. Aitken, "On interpolation by iteration of proportional parts, without the use of differences." Edinburgh Math. Soc., *Proc.*, ser. 2, v. 3, 1932, p. 56–76.

[2] E. H. Neville, "Iterative interpolation." Indian Math. Soc., *Jn.*, v. 20, 1933, p. 87–120.

[3] W. E. Milne, *Numerical Calculus*. Princeton University Press, 1949, chap. III.

[4] J. R. Womersley, "Scientific computing in Great Britain." *MTAC*, v. 2, 1946, p. 110–117.

[5] W. M. Kincaid, "Solution of equations by interpolation." *Annals of Math. Statistics*, v. 19, 1948, p. 207–219.

[6] NBSMTP, *Tables of Lagrangian Interpolation Coefficients*. New York, Columbia University Press, 1944.

[7] E. T. Whittaker & G. Robinson, *The Calculus of Observations*. Fourth ed., London, 1948, chap. IV.

RECENT MATHEMATICAL TABLES

1144[F].—J. Touchard, "On prime numbers and perfect numbers," *Scripta Math.*, v. 19, 1953, p. 35–39.

This note contains a small table of the function

$$C_p(n) = \sum_{k=1}^{n-1} k^p \sigma(k)\sigma(n-k)$$

for $p = 0(1)3$ and $n = 2(1)11$. Here $\sigma(k)$ denotes the sum of the divisors of k. These coefficients occur in the expansion of certain elliptic functions. More specifically,

$$\sum_{n=1}^{\infty} C_p(n)x^n = \Phi_{p-1,\,p}\Phi_{0,\,1}$$

where

$$\Phi_{r,\,s} = \sum_{n,\,m=1}^{\infty} m^r n^s x^{mn}.$$

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1145[I].—E. L. Kaplan, "Numerical integration near a singularity," *Jn. Math. Phys.*, v. 31, 1952, p. 1–28.

The types of singularities considered in this paper are of the forms

$$x^{\frac{1}{2}}A(x), \quad x^{-\frac{1}{2}}A(x), \quad x^n\{A(x)\log x + B(x)\}, \quad n = 0, 1,$$

where A and B are regular near the origin. There are 14 principal tables of coefficients ranging from 3 to 6-term formulas. The accuracy is from 10 to 14D. Certain auxiliary tables of elementary symmetric functions of sets of 2, 3, or 4 small integers are included. The paper suffers from lack of illustrative material. The same problem for $x^{\frac{1}{2}}A(x)$ has been considered by Y. L. LUKE [*MTAC*, v. 6, p. 215–219], who discusses also the paper under review.

D. H. L.

1146[I].—TAKAHIKO YAMANOUCHI, *Tables of Coefficients of Everett's Interpolation Formula*. Report no. 1, Computation Institute, Tokyo, 1946, 19 p. 15 × 21 cm.

These tables give coefficients A and B for Everett's interpolation formula

$$y(x_0 + \theta h) = \theta y_1 + \phi y_0 - (A\delta^2 y_1 + B\delta^2 y_0)$$

where

$$A = \tfrac{1}{6}\theta(1 - \theta^2) \quad \text{and} \quad B = \tfrac{1}{6}\phi(1 - \phi^2), \quad \phi = 1 - \theta,$$

and for the inverse interpolation formula

$$\theta_r = \eta + \alpha A(\theta_{r-1}) + \beta B(\theta_{r-1}),$$

where

$$\eta = \theta_0 = (y - y_0)/(y_1 - y_0), \quad \alpha = \delta^2 y_1/(y_1 - y_0),$$
$$\beta = \delta^2 y_0/(y_1 - y_0) \quad \text{and} \quad x = x_0 + \theta_s h.$$

The coefficients A and B are tabulated for $\theta = 0(.001)1$ [also $\phi = 1(-.001)0$] to 5D, with differences and proportional parts.

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1147[K].—R. A. BRADLEY & M. E. TERRY, "Rank analysis of incomplete block designs. I. The method of paired comparisons," *Biometrika*, v. 39, 1952, p. 324–345.

Given t-treatments with true non-negative preference ratings, π_i and estimates, p_i; $(\sum\pi_i = \sum p_i = 1)$. The treatments are compared in blocks of two, $\binom{t}{2}$ blocks constituting a basic design; n repetitions of the basic design are considered. A likelihood test statistic (B) is given to test the null hypothesis, $H_0: \pi_i = 1/t$, against alternative hypotheses in which m groups of treatments can be formed so that the treatments differ from group to group but are alike within groups. Two special cases are considered: (i) $m = t$, (ii) $m = 2$.

Tables (Appendix A) are prepared to make significance tests for case (i), when $t = 3$ and 4. For $t = 3$, $n = 1(1)10$ and for $t = 4$, $n = 1(1)6$. These tables are in terms of all possible combinations of rank sums for each treatment, where r_{ijk} (= 1 or 2) is the rank of the i-th treatment in the k-th repetition of the block in which both treatments i and j appear. Values of the following are given for each combination of rank sums: p_i to 2D for $i = 1(1)t$; the test statistic, B_1 to 3D; the significance probability, P (probability of this or a smaller value of B_1) to 4D.

A combined analysis is also considered for g groups of rank sums in which the π's may change from group to group, so that it is not desirable to pool

the rank sums. The test statistic B_1^c is found by adding values of B_1 for the individual groups. Appendix B gives values of B_1^c to 4D and P to 3D for $n = n'g$, n', $g = 2(1)5$.

The authors also set up a test of agreement of the rank sums from group to group, and large sample tests for this agreement and for B_1, B_2 and B_1^c. An example is presented for $t = 3$, $g = 2$, $n' = 5$.

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1148[K].—H. E. DANIELS, "The covering circle of a sample from a circular normal distribution," *Biometrika*, v. 39, 1952, p. 137–143.

If a sample of n observations is taken from the circular normal distribution

$$dF = (2\pi\sigma^2)^{-1} \exp[-(x^2 + y^2)(2\sigma^2)^{-1}]\, dxdy$$

the covering circle is defined as the least circle (radius r and centered at distance ρ from the mean) such that all the observations are on or within the circle. The joint distribution of r and ρ is given by

$$dF_n(r, \rho) = -n(n-1)\sigma^{-2}\{\exp[-(r^2 + \rho^2)\sigma^{-2}]\}r[P(r, \rho)]^{n-2}\, drd\rho$$

where

$$P(r, \rho) = \{\exp[-\rho^2(2\sigma^2)^{-1}]\} \int_0^r s\sigma^{-2}\{\exp[-s^2(2\sigma^2)^{-1}]\} I_0(s\rho\sigma^{-2})\, ds$$

and

$$I_\nu(z) = (2\pi)^{-1} \int_0^{2\pi} [\exp(z \cos\vartheta)] \cos \nu\vartheta\, d\vartheta.$$

The distribution for r taken singly is found to be

$$F_n(r) = n\{1 - \exp[-r^2(2\sigma^2)^{-1}]\}^{n-1} - (n-1)\{1 - \exp[-r^2(2\sigma^2)^{-1}]\}^n.$$

A table is exhibited showing the mean to 3D, the variance to 4D, and the 1%, 5%, 95% and 99% points to 3D of the distribution of r/σ for samples of $n = 2(1)15, 20(10)50, 100$.

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1149[K].—H. A. DAVID, "Upper 5 and 1% points of the maximum F-ratio," *Biometrika*, v. 39, 1952, p. 422–424.

The ratio of the largest to the smallest of k sample variances, all with the same number of degrees of freedom ν, was introduced by Hartley[1] as a test statistic for the hypothesis that the population variances are equal, under the assumption of normality. Hartley gave a table of the upper 5% points for this ratio, containing values which were exact for $k = 2$ but were obtained from an approximate expression for $k > 2$. In the present paper numerical quadrature is used to evaluate the exact upper 5% and 1% points for this ratio with an accuracy which may possibly leave the third digit in doubt. Tables thus computed are presented for $k = 2(1)12$, $\nu = 2(1)10, 12,$

15, 20, 30, 60, ∞. It is found that Hartley's approximate table tended to underestimate the 5% points.

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[1] H. O. HARTLEY, "The maximum F-ratio as a short-cut test for heterogeneity of variance," *Biometrika*, v. 37, 1950, p. 308–312 [*MTAC*, v. 5, p. 145].

1150[K].—M. H. GORDON, E. H. LOVELAND, & E. E. CURETON, "An extended table of chi-square for two degrees of freedom, for use in combining probabilities from independent samples," *Psychometrika*, v. 17, 1952, p. 311–316.

These tables give the value of chi-square for two degrees of freedom corresponding to a given probability value. The values of chi-square are given to 4D for the argument, $p = 0(.001).999$. The major function of these tables, according to the authors, is to convert probabilities obtained from independent samples into corresponding values of chi-square so that they may be combined by the technique proposed by Fisher if one is interested in making a new test of significance of the combined data. A numerical illustration showing the application of the Fisher process and the use of the tables accompanies the article.

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1151[K].—E. J. GUMBEL, J. ARTHUR GREENWOOD, & D. DURAND, "The circular normal distribution: theory and tables," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 131–152.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote the angles determined by n points on the circumference of a unit circle, with respect to a fixed radius, and let α_0 denote the angle determined by their centroid, whose radius vector is $\bar{a}$. Then α_0 is a species of mean given by $\sum_{i=1}^{n} \sin (\alpha_i - \alpha_0) = 0$. In this paper the probability density function $f(\alpha)$, for which the maximum likelihood estimate of its location parameter is a mean of the form α_0, is found to be $f(\alpha) = \{\exp [k \cos (\alpha - \alpha_0)]\}/[2\pi I_0(k)]$, where I_0 is the Bessel function of the first kind of purely imaginary argument. This is analogous to the ordinary normal distribution in the case where the n points are situated along a straight line instead of a circle. Table 2 gives the maximum likelihood estimate of the scale parameter k determined from the relation $I_0'(k) - \bar{a}I_0(k) = 0$, for the radius vector $\bar{a}$ for the given observations. Values of k are given to 5D for $\bar{a} = .00(.01).87$ with second central differences that are modified when $\bar{a} \geq .69$.

Table 3 gives 3D values for the function $\psi(\alpha) = \{[\exp (k \cos \alpha)]/I_0(k)\}^{\frac{1}{2}}$ for $k = 0(.1)4$ and $\alpha = 0°(10°)180°$. This function is used in equi-areal (i.e., area-preserving) plotting of the circular normal distribution $f(\alpha)$.

Table 4 gives 5D values of $\Phi(\alpha) = \int_{-\alpha}^{\alpha} f(x)dx$ for $\alpha = 5°(5°)180°$, $k = 0(.2)4$, together with the second central differences in each direction,

δ_a^2, δ_k^2. This table may be used for finding areas or significance levels. It is an abridgment of a 7D table, with intervals half as large, that was computed from the Fourier cosine series for exp ($k \cos x$), and from tables of $I_n(k)$ to $n = 14$ obtained (by arrangement) from the proof sheets of Part II of the BAASMTC's *Table of Bessel Functions*.[1]

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[1] BAASMTC, *Bessel Functions*, Part II. Cambridge, 1952 [*MTAC*, v. 7, p. 97].

1152[K].—A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population from a censored sample," *Biometrika*, v. 39, 1952, p. 260–273.

The author recognizes two ways of censoring a sample: (1) observations above or below a given (truncation) point may be censored (type I); and (2) the $n - k$ smallest or greatest observations may be censored (type II). He derives maximum-likelihood equations for estimating the mean and standard deviation of a normal population from a type II censored sample. Although differing in algebraic form, the author's results when applied to any given sample with the largest (or smallest) measured sample observation, x_k, taken as the truncation point, lead to identical estimates as corresponding estimating equations derived earlier by STEVENS,[1] HALD,[2] and the reviewer.[3] The author indicates that samples considered in the above references are of type I since there, the truncation point is assumed known prior to collecting sample data. It must, however, be pointed out that estimating equations in each of these papers were derived with the total number of sample observations, n, the number of measured observations, k, and hence the number of censored observations, $n - k$, considered as known constants.

Tables are provided in this paper for computing estimates when the right (larger) tail of the sample is censored. The function, $z(\psi, p)$, used in the estimation process, is tabulated to 4D for $\psi = .05(.05).95$ and for $p = .1(.1)1.0$, where

$$\psi = s^2/(s^2 + d^2) = (1 + \eta z - z^2)/(1 + \eta z)$$

and

$$z = \eta + \left(\frac{1}{p} - 1\right)\{\exp(-\eta^2/2)\}\left(\int_z^\infty \exp(-t^2/2)\,dt\right)^{-1}$$

in which s^2 is the sample variance, $d = x_k - \bar{x}$, x_k is the largest measured observation, $\bar{x}$ is the sample mean, $p = k/n$, and $\eta = (x_k - \mu)/\sigma$, where μ and σ are the population mean and standard deviation. For computing asymptotic variances and covariances of the estimates, tables of σ_{ij} ($i, j = 1, 2$) are given to 5D for $p = .05(.05).95(.01).99$, where $[\sigma_{ij}] = [\nu_{ij}]^{-1}$ in which ν_{ij} are elements of the maximum likelihood variance-covariance matrix.

Best (minimum variance) linear estimates, μ^* and σ^* from type II samples for small n are also considered in this paper, and a table with entries to 5D of the coefficients β_i is provided, where

$$\mu^* = \sum_{i=1}^{k} \beta_i x_{i|n}, \quad k = 2(1)n - 1; \; n = 3(1)10$$

in which the $x_{i|n}$ are the x_i, $i = 1, \cdots, k$, rearranged so $x_{1|n} < x_{2|n} < \cdots < x_{k|n}$. A similar table with entries also to 5D is given of the coefficients γ_i, where

$$\sigma^* = \sum_{i=1}^{k} \gamma_i x_{i|n}, \quad k = 2(1)n;\ n = 2(1)10.$$

Two final tables with entries to 5D are given of the variances of estimates σ^* and μ^* for $n, k = 2(1)10$.

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[1] W. L. STEVENS, "The truncated normal distribution" (Appendix to paper by C. I. BLISS, "The calculation of the time mortality curve"), *Ann. Appl. Biol.*, v. 24, 1937, p. 815–852.

[2] A. HALD, "Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 119–134. *MTAC*, v. 5, p. 74.

[3] A. C. COHEN, JR., "Estimating the mean and variance of normal populations from singly truncated and doubly truncated samples," *Ann. Math. Stat.*, v. 21, 1950, p. 557–569.

1153[K].—M. R. SAMPFORD, "The estimation of response-time distributions. II. Multi-stimulus distributions," *Biometrics*, v. 8, 1952, p. 307–369.

"The types of response time distribution occurring when two or more stimuli act on a sample of individuals are discussed. Two situations are discussed in considerable detail; the 'accidental death' model, in which each response can be related to its appropriate stimulus, and the 'natural death' model in which the exact cause of any death cannot be determined, but the distribution of 'potential survival times' to the two stimuli can be assumed bivariate normal. Maximum likelihood methods of estimation are developed for these situations, and tables are given to simplify the calculations." (From the author's summary.) Tables of ν, λ, and ξ are given to 5D for $\eta = -5(.1) - 3(.01)3(.1)5$, where $Z(\eta) = (2\pi)^{-\frac{1}{2}} \exp(-\eta^2/2)$, $Q(\eta) = 1 - \int_{-\infty}^{\eta} Z(u)du$, $\nu = Z(\eta)/Q(\eta)$, $\lambda = \nu(\nu - \eta)$, and $\xi = \nu - \eta\lambda$. It should be noted that ξ is always positive. In the tables a curious misprint occurs for all $\eta \geqq -2.75$ in attributing to ξ a minus sign.

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1154[K].—M. E. TERRY, "Some rank order tests which are most powerful against specific parametric alternatives," *Annals Math. Stat.*, v. 23, 1952, p. 346–366.

A statistic $c_1(R)$ based on a method of HOEFFDING[1] yields a most powerful rank order test of H_0: two samples of m and n observations come from the same continuous population, against the alternative H_1: the two samples come from two normal populations with common variance σ^2, and with means θ and ϕ, respectively, where $(\theta - \phi)/\sigma$ is positive and small. An

approximation to the null distribution of c_1 in terms of the t-distribution and a comparison of the exact and approximate critical values for $N = 6(1)10$ for certain m and n is given. Under certain conditions, the asymptotic distribution of c_1 under H_0 is shown to be normal. Table 1 gives the exact null distribution of $c_1(R)$ to 2D for $N = 2(1)10$ for $m \leq n$, $m + n = N$ and distinct permutations of R (i.e. permutations of m 0's and n 1's), together with the corresponding values of the MANN-WHITNEY[2] U test. Table 2 gives the probability p of exceeding the critical values c_1 under H_0 for all $p \leq .1$ to 3D for $N = 6(1)10$.

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[1] W. HOEFFDING, " 'Optimum' nonparametric tests," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Univ. of California Press, Berkeley, Calif., 1951, p. 83–92.

[2] H. B. MANN & D. R. WHITNEY, "On a test of whether one of two random variables is stochastically larger than the other," *Annals of Math. Stat.*, v. 18, 1947, p. 50–60.

1155[K].—J. E. WALSH, "Operating characteristics for tests of the stability of a normal population," Am. Stat. Assn., *Jn.*, v. 47, 1952, p. 191–202.

Let n independent observations be drawn from a normal population with unknown mean m' and unknown standard deviation σ'. The problem is to test whether a small sample came from a specified normal population with known mean m and known standard deviation σ. In this situation, the null hypothesis tested is that $m' = m$ and $\sigma' = \sigma$.

The most useful result of the paper is a set of tables which give the operating characteristic (OC) function values for the following three types of tests. The first test, and the most common, is based on the "limits" for the sample mean, $\bar{x}$, and the standard deviation, s (using $n - 1$). In the second test, let $\bar{x}$ be replaced by the t-statistic

$$t = n^{\frac{1}{2}}(\bar{x} - m)/s,$$

so that the limits for t and s are used. In the third test, the statistic s is replaced by

$$s(m) = \left[\sum_{i=1}^{n} (x_i - m)^2/n\right]^{\frac{1}{2}}$$

to give a test based on the limits of $\bar{x}$ and $s(m)$.

Table 1 provides "limits," given to 3D, which define the critical region for t and for s at $\alpha = .005$, $.01(.01).05$ and $n = 3, 5$. Table 2 provides "limits," given to 3D, which define the critical region for $\bar{x}$ and for $s(m)$ at the same values of α and n as given in Table 1. If one or both of the statistics fall outside these limits, the null hypothesis is rejected.

The OC function for a test is defined to be the probability that both statistics have values within their limits given the true values of m' and σ'. However, in this paper, the OC function values are given for alternative hypotheses which are expressed in terms of two equivalent parameters a and b, where

$$a = n^{\frac{1}{2}}(m - m')/\sigma', \quad b = \sigma/\sigma'.$$

Table 3 contains an OC function comparison, given to 3D, for the three types of tests at values of $\alpha = .01$; $n = 3, 5$; $a = 0(1)4$; $b = 1/8, 1/4, 1/2, 1, 2, 4, 8$. Table 4 contains the OC function values, given to 3D, for the test based on $\bar{x}$ and s at values of $\alpha = .001, .005, .01, .02, .05$; $n = 3, 5, 7$; $a = 0(1)4$; $b = 1/8, 1/4, 1/2, 1, 2, 4, 8$.

The results of this paper are directly applicable to those quality control situations for which the assumptions underlying the test are satisfied. This reviewer would enjoy the extension of the OC function values to the $\bar{x}$ and R test together with the three types of tests included in this paper.

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1156[L].—S. Chandrasekhar & Donna Elbert, "The roots of $J_{-(l+\frac{1}{2})}(\lambda\eta)J_{l+\frac{1}{2}}(\lambda) - J_{l+\frac{1}{2}}(\lambda\eta)J_{-(l+\frac{1}{2})}(\lambda) = 0$," Cambridge Phil. Soc., *Proc.*, v. 49, 1953, p. 446–448.

The authors tabulate to 6S the first root, λ_1, of the equation mentioned in the title, together with 6S or 7D values of

$$J_{\pm(l+\frac{1}{2})}(\lambda_1), \quad J_{\pm(l+\frac{1}{2})}(\lambda_1\eta), \quad \frac{2}{(\lambda_1\pi)^2}\left(\frac{J^2_{l+\frac{1}{2}}(\lambda_1\eta)}{J^2_{l+\frac{1}{2}}(\lambda_1)} - 1\right)$$

for $\eta = .2$, $l = 1(1)5$; $\eta = .3$, $l = 1(1)6$; and $\eta = .4, .6, .8$, $l = 1(1)15$.

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1157[L].—W. J. Duncan, *Normalised Orthogonal Deflexion Functions for Beams*. Aeronautical Research Council *Reports and Memoranda*, No. 2281. His Majesty's Stationery Office, London, 1950. 23 p.

The orthogonal deflexion functions are

$$S_n(\xi) = \frac{(4n+1)^{\frac{1}{2}}}{(2n)!}\frac{d^{2n-2}}{d\xi^{2n-2}}[\xi^{2n}(1-\xi)^{2n}],$$

$$A_n(\xi) = -\frac{(4n+3)^{\frac{1}{2}}}{(2n+1)!}\frac{d^{2n-1}}{d\xi^{2n-1}}[\xi^{2n+1}(1-\xi)^{2n+1}],$$

and can be shown to be numerical multiples of $(1-x^2)^2P_m''(x)$ where $x = 1 - 2\xi$, $m = 2n$ for S_n, $m = 2n+1$ for A_n, and P_m is the Legendre polynomial.

The Green's functions for a doubly built-in uniform beam are

$$G_1(\xi, \tau) = \tfrac{1}{6}\xi^2(1-\tau)^2[3\tau - \xi(1+2\tau)], \quad 0 \leq \xi \leq \tau \leq 1,$$

$$G_1(\xi, \tau) = G_1(\tau, \xi), \quad G_2(\xi, \tau) = \frac{\partial}{\partial\tau}G_1(\xi, \tau).$$

Tables 1 to 5 were computed under the direction of R. A. Fairthorne, and give 5D values of S_1, S_2, S_3, A_1, A_2 for $\xi = 0(.01)1$.

Tables 6 and 7 were computed by S. Kirkby and give 5D values of G_1 and G_2 for $\xi, \tau = 0(.05)1$.

A. E.

1158[L].—COSTANTINO FASSO, "Di un integrale intervenuto in una questione di idraulica," Ist. Lombardo Sci. Lett., *Rend.*, *Cl. Mat. Nat.*, s. 3, v. 15, 1951, p. 471–497.

The author denotes by $\Phi(x, z)$ a suitably defined indefinite integral of

$$\frac{x^5}{z + x^5}$$

with respect to x. He gives 5D values of $\Phi(x, z)$ for $x = .86, .9(.02)1.1, 1.14$ and $z = 10, 5, 2.5, 1.5, 1, .5, .25, .1, 0$ and for fewer x values and $-z = .25, .5, .7, 1, 1.2, 1.5, 2.5, 5, 10$. He also gives some numerical data and diagrams of the level curves of the surface,

$$y = \Phi(x, z) - \Phi(1, z)$$

in (x, y, z) space.

A. E.

1159[L].—MARION C. GRAY, "Legendre functions of fractional order," *Quart. Appl. Math.*, v. 11, 1953, p. 311–318.

The author gives a 6D table of $P_\nu(\cos\theta)$ for $\nu = .1(.1)2$, $\theta = 10°(10°)170°, 175°$. In comparison with the tables described in RMT **1110**, the present table is much shorter in the ν direction, much thinner in the θ direction, and there are no associated Legendre functions; but on the other hand there is one more decimal, and the tabulation extends to $\theta = 175°$ (instead of $\theta = 90°$).

There is a discussion of the zeros of $P_\nu(\cos\theta)$ for fixed θ and variable ν (see also RMT **1120** and the literature quoted there), and there are several useful expansions for numerical computation.

A. E.

1160[L, V].—J. BARKLEY ROSSER & R. J. WALKER, *Properties and Tables of Generalized Rocket Functions for Use in the Theory of Rockets with a Constant Slow Spin.* Cornell University, 1953, 21.5 × 28 cm.

This report treats certain indefinite integrals that describe the motion of fin-stabilized rockets having a constant slow spin about the axis of symmetry. The integrands are products of trigonometric functions and Fresnel integrals. In terms of functions defined in *MTAC*, vol. 2, p. 213 and *MTAC*, v. 3, p. 474, the integrals tabulated are

$$Rrc(\alpha, x) = \int_x^\infty Rr(w)\cos\alpha(w - x)\,dw$$

and the similar integrals obtained when $Rr(w)$ is replaced by $Ri(w)$ and when cos is replaced by sin. About half the space is devoted to a tabulation of these four integrals to 5D for $x = 0.0(0.1)5.0$ and $\alpha/\pi = 0.1(0.1)2.0$. For a lesser range of x, the tables extend over $\alpha/\pi = 2.0(0.1)8.0$. Second differences with "throw-back" are given for x. Auxiliary formulas and tables enable the entire range of real α and x to be covered. Careful attention is given to the accuracy attainable in the various regions. About half the space is devoted to definitions of the various rocket functions, to their connection with differential equations, to a review of the properties of the

simpler *Rr* and *Ri* functions referred to above, to a development of the properties of the functions tabulated, and to the methods of computation of the tables.

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1161[L].—TATUDIRO SASAKI, *Tables of* $\mu(x) = R\psi(ix)$ *and* $\sigma_0(x) = \arg\Gamma(1-ix)$. Numerical Computation Bureau, Tokyo, 1950. 9 pages, mimeographed.

These tables contain values of $\mu(x) = \mathrm{Re}\,\psi(ix)$ and $\sigma_0(x) = \arg\Gamma(1 - ix)$ where x is a positive variable, $\Gamma(z)$ is the gamma function and $\psi(z)$ is its logarithmic derivative. The values of $\mu(x)$ are tabulated to 6D with second central differences for $x = 0(.01).5(.02)2(.05)2.5$. The table of $\sigma_0(x)$ is also to six decimal places with second central differences for $x = 0(.01)3$. The auxiliary function $S_0(y)$ defined by the relation $-\sigma_0(x) = x \ln x - x + S_0(y)$, $y = 1/x$ is given to 6D for $y = 0(.01).35$ with first differences.

Comparison with the tables of the same functions published by NBSCL[1] showed a transposition in the value for $x = .30$ in the table of $\mu(x)$ at the top of page 1. There is also a gradually increasing discrepancy of one to four units in the values of $\sigma_0(x)$ from $x = 2.40$ to $x = 3.00$.

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NBSCL

[1] NBSCL, *Tables of Coulomb Wave Functions*, v. 1. AMS no. 17, Washington, 1952 [*MTAC*, v. 7, p. 101–102].

1162[L].—L. J. SLATER, "On the evaluation of the confluent hypergeometric function," Cambridge Phil. Soc., *Proc.*, v. 49, 1953, p. 612–622.

This paper contains an 8S table of

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{\Gamma(b)\Gamma(a+n)x^n}{\Gamma(a)\Gamma(b+n)n!}$$

for $a = -1(.1)1$, $b = .1(.1)1$, $x = 1(1)10$, and the asymptotic expansion of this function for large x with converging factors which improve the accuracy quite considerably.

The tables were computed from the power-series expansions, about thirty terms of the series being required. The computations were carried out on the EDSAC at the Cambridge Mathematical Laboratory.

A. E.

1163[L].—UNIVERSITY OF CALIFORNIA, Department of Meteorology, *Tables relating to Rayleigh scattering of light in the atmosphere*. Computed for the project *Investigation of polarization of sky light* by direction of ZDENEK SEKERA, Project Director. Computations performed under supervision of GERTRUDE BLANCH, by the U. S. Department of Commerce, NBS, Los Angeles, INA. November, 1952, xxvi + 85 p., 1 p. Errata. 21 × 27 cm.

This is a collection of nine tables useful in the study of the polarization of the light in the earth's atmosphere. The numerical work presented here

is based on the theory developed by S. CHANDRASEKHAR,[1] who in 1947 found the exact solution to this classical problem. A helpful feature of the publication is the careful definition of all tabulated quantities as well as a description of their physical significance. The fundamental functions on which the solution of the whole problem depends, are the four pairs of functions $X^{(k)}(\mu, \tau)$ and $Y^{(k)}(\mu, \tau)$ $(k = 1, 2, 3, 4)$ defined by the simultaneous integral equations

$$(1)\qquad \begin{aligned} X^{(k)}(\mu) &= 1 + \mu \int_0^1 [X^{(k)}(\mu)X^{(k)}(t) - Y^{(k)}(\mu)Y^{(k)}(t)] \frac{\psi_k(t)}{t+\mu}\, dt, \\ Y^{(k)}(\mu) &= e^{-\tau/\mu} + \mu \int_0^1 [Y^{(k)}(\mu)X^{(k)}(t) - X^{(k)}(\mu)Y^{(k)}(t)] \frac{\psi_k(t)}{\mu - t}\, dt \end{aligned}$$

where

$$\psi_1(t) = \tfrac{3}{8}(1 + t^2 - 2t^4), \quad \psi_2(t) = (1 + 2t^2 + t^4),$$
$$\psi_3(t) = \tfrac{3}{4}(1 - t^2), \quad \text{and} \quad \psi_4(t) = \tfrac{3}{8}(1 - t^2).$$

Successive approximations $X_n^{(k)}(\mu, \tau)$ and $Y_n^{(k)}(\mu, \tau)$ to the solutions of (1) were obtained by an iteration procedure, starting with the values

$$X_0^{(k)}(\mu, \tau) = 1, \quad Y_0^{(k)}(\mu, \tau) = e^{-\tau/\mu}.$$

All calculations were made with IBM machines. Depending on the value of τ, six or seven iterations were carried out. Table I contains the values of $X_n^{(k)}(\mu, \tau)$ and $Y_n^{(k)}(\mu, \tau)$ to 5D,

for $\tau = 0.15$ and 0.25, $n = 6$, $\mu = 0(.02).32(.04).96(.02)1$
and for $\tau = 1$, $n = 7$, $\mu = 0(.02).2(.04).96(.02)1$.

The intervals used in the numerical integrations were such that the entries are guaranteed only to within 0.0001 for $\mu = 0.02$, and to within 0.00003 for $\mu \geq 0.04$. In order to exhibit the behavior of successive approximations to X and Y, in Table IX representative values are listed to 7D. Since for $k = 3$ the solutions of (1) are not unique, the functions tabulated (called "standard solutions") are those defined by the relations

$$X^{(s)} = X^{(3)} + q\mu[X^{(3)} + Y^{(3)}]; \quad Y^{(s)} = Y^{(3)} - q\mu[X^{(3)} + Y^{(3)}]$$

where

$$q = y_0^{(3)}/[x_1^{(3)} + y_1^{(3)}],$$

$x_j^{(k)}$ and $y_j^{(k)}$ being the "moments" of the X and Y functions:

$$x_j^{(k)} = \int_0^1 \psi_k(t) X^{(k)}(t, \tau) t^j\, dt,$$
$$y_j^{(k)} = \int_0^1 \psi_k(t) Y^{(k)}(t, \tau) t^j\, dt.$$

Using the various formulas given by the theory, the authors of this publication finally illustrate, in Table VIII, the predicted behavior of the

sky radiation by considering the intensity and polarization on the principal meridian (containing the sun) for various elevations of the sun. The corrections to the intensity and polarization predicted by the theory for a ground surface reflecting according to Lambert's law with albedos $\lambda = 0.10, 0.25, 0.50$, and 0.80, have also been included in Table VIII.

The tables presented here not only provide the numerical results predicted by an exact theory of atmospheric scattering according to Rayleigh's laws, but also provide a basis for a quantitative evaluation of the non-molecular component of the sky radiation.

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[1] S. CHANDRASEKHAR, *Radiative Transfer*. Oxford, 1950.

1164[L].—KLAUS ZWEILING, *Grundlagen einer Theorie der biharmonischen Polynome*. Verlag Technik, Berlin, 1952, viii + 130 p., 5 plates, 1 insert, 18 × 24 cm.

The polynomials tabulated in this volume are

$$P_{0,0} = 1, \quad P_{1,0} = x, \quad P_{1,1} = xy$$

$$P_{2n,0} = \tfrac{1}{2}x(w^{2n-1} + \bar{w}^{2n-1}) \qquad n = 1(1)6,\ w = x + iy,\ \bar{w} = x - iy$$

$$P_{2n,1} = \frac{x(w^{2n} - \bar{w}^{2n})}{4ni} \qquad n = 1(1)5$$

$$P_{2n+1,0} = \frac{1}{4n}\left[2nx(w^{2n} + \bar{w}^{2n}) - iy(w^{2n} - \bar{w}^{2n})\right] \qquad n = 1(1)5$$

$$P_{2n+1,1} = -\frac{y(w^{2n+1} + \bar{w}^{2n+1}) + i\,(2n+1)x(w^{2n+1} - \bar{w}^{2n+1})}{4n(2n+2)} \qquad n = 1(1)5$$

$$p_{11,1} = x^{-3}y^{-1}P_{11,1}, \quad p_{12,0} = x^{-2}P_{12,0}, \quad p_{12,1} = x^{-2}y^{-1}P_{12,1},$$

all for $x, y = 0(.1)1$.

The zero lines of these polynomials are also tabulated, graphs of the $P_{i,j}$ are given, as are explicit representations in Cartesian and polar coordinates, various formulas and properties, and the expression of $\int P_{m,n}P_{i,k}\,dx$ in terms of products of these polynomials, for $m + n,\ i + k \leq 6$.

The polynomials tabulated here are useful in problems concerning the bending of elastic plates. Similar quantities have also been tabulated by THORNE.[1]

A. E.

[1] C. J. THORNE, "A table of harmonic and biharmonic polynomials and their derivatives," Utah Engineering Experiment Station, *Bulletin*, no. 39, 1949, supplement. Salt Lake City.

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **1153, 1161,** and in NOTE **158.**

233.—LE CENTRE NATIONAL D'ÉTUDES DES TÉLÉCOMMUNICATIONS, *Tables des fonctions de Legendre associées*. Paris, 1952. [*MTAC*, v. 7, p. 178].

The authors report that on p. 17 the entry for $n = 3.1$, $\theta = 57°$ should read -0.4259087.

A. E.

234.—L. S. KHRENOV, *Semiznachnyĕ Tablit͡sy Trigonometricheskikh Funkt͡siĭ.* (Seven-figure tables of trigonometric functions.) 1951 [*MTAC*, v. 7, p. 238–239].

In Table II, p. 131–401, the following errata were discovered by inspection and checking of differences.

Page	*Angle*	*Function*	*For*	*Read*
153	3 34 50	csc	01284	012*34*
176	7 26 30	tan	6160	616*9*
196	10 47 0	sec	78757	7*9*757
202	11 41 40	ctg	1181	11*79*
205	12 15 10	tan	1711	17*2*1
	20	tan	2218	22*2*8
	30	tan	2726	27*3*6
	40	tan	3234	32*4*4
209	59 30	csc	835	*9*35
210	13 2 20	csc	828	*9*28
219	14 35 10	ctg	3878	*2*878
220	44 10	csc	726	72*4*
250	19 45 30	sin	466	4*5*6
251	51 20	sin	466	4*5*6
277	24 15 50	ctg	8461	84*8*1
278	23 10	sec	9574	95*5*4
287	25 52 20	ctg	256	25*4*
300	28 7 50	sin	527	*4*27
313	30 11 40	ctg	86553	8*5*553
320	31 29 20	ctg	1778	177*6*
336	34 3 40	csc	54683	546*9*3
343	35 12 40	ctg	1.4160069	0069
	50	ctg	8611	*1.416*8611
	18 20	cos	279	2*81*
347	58 10	csc	1136	113*8*
376	40 43 20	ctg	6966	69*5*6
382	41 40 30	cos	9224	92*8*4

Twenty values in which the errors amounted to one unit in the last place have not been included in this list.

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New York 25, N. Y.

235.—NBS Applied Mathematics Series, No. 6, ***Tables of the Binomial Probability Distribution.*** Issued January 1950; reprinted October 1952.

The following errors occur in one or both of the printings: errors marked with an asterisk occur in both editions; those without an asterisk were corrected in the reprinted edition.

				Entry	
Page	n	r	p	*for*	*read*
* 7	8	3	.06	.0086873	.0088773
18	14	2	.18	.2624913	.2724913
116	37	0	.09	.0395163	.0305163
134	40	3	.07	.2211640	.2311640
134	40	4	.07	.1709448	.1609448
140	41	5	.06	.0668162	.0628162
140	41	6	.06	.0200573	.0240573
192	49	17	.35	.0289183	.1189183
192	49	18	.35	.2038364	.1138364
200	6	3	.47	.6984534	.5984534
*212	13	3, 4, 5, 6; interchange values for $p = .14$ and $p = .15$			
244	24	22	.45	.0000001	.0000021
*319	39	19	.49	.5_68882	.5768882
327	40	4	.07	.3163132	.3063132
384	49	18	.35	.5424066	.4524066

It should be noted that several corrigenda may have resulted from a single error in computation or transcription. For example, the transposition of two digits on page 384 generated two further errors on page 192.

We are indebted to many users of the tables for reporting these errors.

Churchill Eisenhart

National Bureau of Standards
Washington 25, D. C.

UNPUBLISHED MATHEMATICAL TABLES

177[A].—J. W. Wrench, Jr., *A New Approximation to* $180\pi^{-1}$. One manuscript page on deposit in the UMT File.

This 2035D approximation is the by-product of the calculation of π^{-1}. [See Note 159 in this issue.]

J. W. Wrench, Jr.

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Washington 16, D. C.

178[F].—A. Ferrier, *Factor table for* $3n^4 - 1$. One photostat page 43×63 cm. Deposited in the UMT File.

This table gives 316 complete factorizations of $3n^4 - 1$ for $n < 1000$. There is also a table giving the values of n modulo p for which $3n^4 - 1$ is divisible by $p = 12k \pm 1 < 3000$.

A. Ferrier

College de Cusset
Allier, France

179[L].—T. H. Crowley, *Tables of Integrals of certain Bessel Functions.* Available at the Antenna Laboratory, Ohio State University, Columbus, Ohio.

These tables give values of the integrals

$$\int_0^u J_0(\lambda x) \sin x \, dx \quad \text{and} \quad \int_0^u J_0(\lambda x) \cos x \, dx$$

for

$$u = 0(.02)10, \quad \lambda = 0(.1)10, \quad \lambda u \leq 15.$$

Although the calculations were designed to give 4D accuracy, spot checking indicates an accuracy of 5D.

T. H. Crowley

Ohio State Univ.
Columbus, Ohio

180[L].—E. W. Pike, *Table of Parameters for the Summation Analogue of Laguerre Polynomials.* Two typewritten pages on deposit in the UMT File.

This table is of use in the design of filters for pulsed information.

E. W. Pike

Raytheon Mfg. Co.
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AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. Cannon, 415 South Building, National Bureau of Standards, Washington 25, D. C.

Technical Developments

INPUT AND ORGANIZATION OF SUB-ROUTINES FOR FERUT

1. Introduction.—Descriptions of methods for handling sub-routines on other machines have been written for the Manchester Electronic Computer,[1] EDSAC,[2] SEAC[3] and ILLIAC[4]. The purpose of this article is to discuss the input and storage of routines and numerical data and the organization of routines during the solution of a problem on Ferut.

Ferut is the serial, one address, electronic digital computer now in operation in the Computation Centre, University of Toronto, Canada. It was built by Ferranti Ltd., Manchester, England, and is similar to the computer at the University of Manchester, England. Ordinary 5-hole telegraphic punched tape is used for input with a photoelectric reader. The output mechanism is a teleprinter and punch.

Information in the machine is kept in blocks or pages. In each page there are 64 short lines of 20 binary digits each. For convenience these 20 bits are arranged in four groups of five bits. The five-bit group constitutes a digit in the scale of 32 and is represented by one teleprint character. An electronic instruction consists of one short line, two teleprint characters for

the address, and two for the order. Numbers consist of two short lines or 40 bits.

The electronic storage comprises eight pages labelled $S_0, S_1, \cdots, S_7$. In addition, there is a magnetic storage of 256 tracks on a drum, each track holding two pages of information. The pages may be transferred singly or in pairs to or from a magnetic track. This is accomplished by a magnetic transfer instruction.

2. Routine Changing.—During the operation of a program, space in the electronic store is at a premium and thus the instructions necessary to carry out a problem are divided, as is customary, into a main, or master routine and a number of sub-routines. Only the set of instructions immediately in use is kept in the electronic store; the other routines are in the magnetic store. When it is desired to change from one routine to the next, it is necessary to have a group of orders which will (1) preserve, in a list called the link list, the location in magnetics of the routine being left, (2) record the line at which it is being left, (3) locate the next routine and call it down from magnetics and finally (4) enter the new routine at the appropriate line. This group of instructions is common to all routine changes and thus is kept separate from any one routine in a *Routine Changing Sequence* (R.C.S.) which is permanently down in the electronic store. (This occupies page S_2 along with powers of 2 for shifting and some other constants which are frequently required, and the page is known as PERM.)

There must be, however, in the routine being left a few instructions supplying the required information about the specific routine to be entered and a control transfer to R.C.S. This linking sequence of orders which appears in the routine being left must be kept to a minimum to conserve space, and in the case of the Ferut system it occupies four short lines. One of these lines is, of course, the magnetic transfer which calls down the next routine from the magnetic store to the electronic store.

This transfer is recorded in the link list by R.C.S. so that it can be used at a later date. The list of links grows as the control moves down a chain of sub-routines changing from the master routine (zero-th level) to the first level sub-routine, to the second level sub-routine, etc. The list is consulted by R.C.S. as the control returns back up the chain of sub-routines to the master routine.

3. Directory.—The magnetic tracks are numbered, and in writing a magnetic transfer it is necessary to specify the track number. For several reasons it may be desirable to change tracks after a problem has been coded and the tapes have been punched. You must then seek out and alter all magnetic transfers which, of course, are embedded in the program. This is not only tedious but leads to errors. To overcome this difficulty all magnetic transfers are kept separate from the routines proper in a list called the directory. In the line of the routine where the transfer should appear is placed the address in the directory of the required magnetic transfer. This line is called a false line. During input the false lines are replaced by the appropriate magnetic transfers from the directory.

If the magnetic transfer required to call down a certain routine is stored in line ab of the directory, ab is said to be the *Directory Number* of the routine. This directory number is what is written in a false line. When a routine has been tested and is considered useful, it may be assigned a

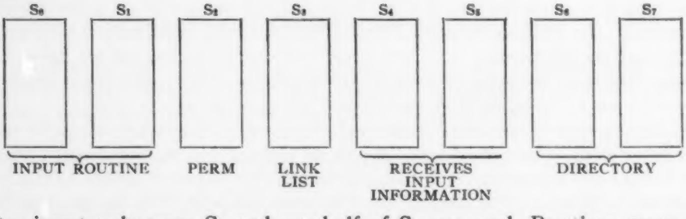
Library Number which will be its permanent directory number. Sixteen lines at the beginning of the directory are kept open for routines of a particular problem. These directory numbers are temporary.

4. Input and Output.—To perform input or output operations it is necessary to combine the simple input or output orders of the machine with other orders in an input or output routine. The function of the Ferut *Input Routine* is to read information from tape, perform certain alterations on routines or numerical data and store the routines or data in assigned locations in the machine.

The input routine reads one character at a time from the tape testing it until it finds a *Warning Character*. There are eleven warning characters. Each of the warning chs. causes control to be taken to a certain section of the input routine where a prescribed group of instructions is obeyed.

An attempt has been made to choose, for warning chs., teleprint characters which are suggestive of the operations which the warning chs. perform. For example—the warning ch. J followed by n/ causes the n chs. following the / to be printed out by the machine. (The ch. J suggests the word JOT.) After the Jn/ the machine reads in n chs. before it begins again to examine the tape for the next warning ch. Since / (zero) is not a warning ch., blank tape is examined rapidly and passed by. Thus between meaningful sequences on the tape (such as the Jn/ followed by n chs.) it does no harm to have spaces, and they are a great assistance when reading the tape by eye.

The use of the electronic pages during input is best described by a diagram.



After input only page S_2 and one half of S_3 are used. Routines normally operate from S_0 and S_1.

The list of warning characters is as follows:

STOP—Stops the tape—control is returned to input to search for the next warning ch. when a prepulse is given manually.

Jn/—Prints out the n chs. on the tape following Jn/—used for titling routines (J suggests JOT).

E ab—Transfers control to line (ab + 1) (E suggests ENTER).

: abcd—Causes abcd to be obeyed as a magnetic instruction (: suggests /: the "obey as a magnetic instruction" order of Ferut).

K ab n—Reads in from the tape n short lines starting the entries in line ab of the store. A maximum of 32 short lines can be read in at a time. (K suggests absolutely nothing but is used for this purpose by all the other input systems.)

U ab—Writes Up / D ab—Reads Down } the routine whose directory number is ab {from / to} page S_4 (or pages S_4 and S_5 if it is a 2 page routine) {to / from} the magnetic track assigned in the directory. Check of the magnetic transfer is performed, and a warning signal indicates failure.

$\left.\begin{matrix}\text{@ abcd}\\ \text{R abcd}\end{matrix}\right\}$—Calls in as an $\left\{\begin{matrix}\text{ad-routine}\\ \text{sub-routine}\end{matrix}\right\}$ the routine whose directory number is ab and enters it at line (cd + 1). (@ sounds like ad—and R is suggestive of Routine.)

F ab—Replaces the false line ab of a routine by the contents of a line of the directory. (Explanation: In order to incorporate a magnetic transfer within a routine in line ab the directory number of the appropriate magnetic transfer is placed in the routine itself in line ab. This is a false line. The warning ch. F replaces this by the appropriate true line, i.e., the magnetic transfer as it appears in the directory. For example: One line of the linking sequence is such a false line and any routine containing a linking sequence must be followed by the proper F sequence.)

C/(2n)—Adds together the 32 long lines of page n and compares the sum modulo 2^{40} with the known *Check Sum* of this page. If the two differ, a signal occurs. The known check sum must, of course, be planted in a certain line of the store.

A wide variety of routines have been devised for handling the input and output of numerical data depending on the position of decimal or binary point, method of conversion, number of significant figures, layout, etc. These are entered as sub-routines from the input routine by an R warning ch. or from the master routine of the problem by a linking sequence. When numerical data are put in, the pages of information can be given directory numbers and thus stored and brought down in a manner similar to that used in handling sub-routines.

The following checks are a part of this system of input. Each Up or Down transfer is checked; thus the contents of magnetic and electronic storage are known to be identical. The transfer from tape to electronic storage is checked by having a known check sum of a routine or of numerical data read in and compared with the check sum of the contents of the electronic page by means of the C warning ch. Alternately, if the check sum is not known, the tape may be read in twice, once with current on which writes from electronic to magnetic storage, the second time with this write current off. The Up transfer check then effectively compares what was read in the first time from tape with what was read in the second time.

Most problem tapes consist of sequences of characters which (i) print the title, (ii) read in and store the directory, (iii) read in, alter and store routines and (iv) start the problem. Titling (see i) is useful for identifying tapes. Process (ii) consists of reading the magnetic transfers for the routines being used into the appropriate lines of the directory. The directory is stored by means of the : warning ch. All routines are read (see iii) by K-sequences into S_4 (or S_4 and S_5 in the case of two-page routines). They are checked if desired by the C ch., altered if necessary, i.e., false lines are replaced by F-sequences, and then stored by the appropriate Up order. The problem is started (see iv) by entering the master routine as an ad-routine by means of an @ sequence on the tape.

Routines may be altered during input by bringing them down with a D-sequence, altering the desired lines by K-sequences and then sending them back up to magnetics by a U-sequence.

Library tapes may be used without alteration in a problem tape. They consist of a titling sequence, K-sequences reading the routine into S_4 or S_4 and S_5 and then the Up order using the library number of the routine. Library tapes have a check sum taken by means of the Down order followed

by the C warning ch. After this false lines are replaced by F-sequences, and the routine is stored again by the U-sequence.

5. Example.—Consider the organization of the routines for the two-dimensional Fourier synthesis

$$\rho(y, z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} F_{kl} \cos 2\pi(ky + lz)$$

where the coefficients F_{kl} are tabulated. The program for this problem will consist of four routines and the data. Three of the routines to be used are library routines, namely, Decimal Input, Decimal Output, and Cosine. The fourth is the Master routine. The latter must call in Cosine and Decimal Output as sub-routines and thus has a false line in each of the linking sequences calling in these two routines. These false lines have addresses a_1b_1 and a_2b_2. Transfers calling down the data require false lines in a_3b_3, a_4b_4, etc.

The coefficients F_{kl} are taken in and converted from decimal to binary form by the Decimal Input routine before they are stored in the magnetics. This is accomplished during input by entering Decimal Input as a sub-routine from the tape by means of the R warning ch. Details for the routines and data are given in Table I.

The magnetic transfer calling down the Master will be $(t_1)\ \delta_2 S_0$, where (t_1) represents the two teleprint character track address, S_0 indicates the electronic storage from which the routine operates, and δ_2 denotes that it is a two-page routine. This magnetic transfer must be recorded in line /N of the directory.

TABLE I

Description	Directory Number	Magnetic Storage	Electronic Storage
Master	/N	t_1	S_0, S_1
Decimal Input	SF	t_2	S_0, S_1
Decimal Output	ON	t_3	S_0, S_1
Cosine	XN	t_4 (left half)	S_0
Data 1 (two pages)	EN	t_5	S_4, S_5
Data 2 (two pages)	@N	t_6	S_6, S_7
Data 3 (two pages)	AN	t_7	S_4, S_5
etc.			

The tape is organized in the following way:

Punching	*Description*
T T T T T	T is a letter which is not a warning ch. but indicates the beginning of the tape to the operator.
spaces	. . . .
JZ / FOURIER: SYNTHESIS	Title (Z is the teleprint character for 17.)
spaces	. . . .
	The K-warning ch. reads into the directory the magnetic transfer for bringing down:
K / N E $(t_1)\ \delta_2\, S_0$	the Master
K S F E $(t_2)\ \delta_2\, S_0$	Decimal Input
K O N E $(t_3)\ \delta_2\, S_0$	Decimal Output
K X N E $(t_4)\ \delta_1\, S_0$	Cosine

Punching	*Description*
K E N E (t_5) δ_2 S_6	Data 1
K@N E (t_6) δ_2 S_6	Data 2
K A N E (t_7) δ_2 S_6	Data 3
etc.	(E is teleprint for 1.)
:I / R N	I / R N is the magnetic transfer instruction which stores the directory on track.
S T O P	
Library tape for Decimal Input Library tape for Decimal Output Library tape for Cosine	these routines all have check sums
Master routine punched with the K-sequences which read it into S_4 and S_5.	
Fa_1b_1 Fa_2b_2 Fa_3b_3 Fa_4b_4, etc.	Replaces false lines in Master by the magnetic transfers from the directory.
U / N	Stores Master on track t_1.
S T O P	
R S F c d Data 1	Enters Decimal Input as a sub-routine at line cd + 1 whereupon Data 1 punched in the appropriate form are read in and converted.
U E N	Stores Data 1 on track t_5.
R S F c d Data 2	Reads in and converts Data 2.
U @ N	Stores Data 2.
R S F c d Data 3	Reads in and converts Data 3.
U A N	Stores Data 3.
etc.	
S T O P	
@ / N c d	Enters the Master as an ad-routine at line cd + 1.
\$ \$ \$ \$	A non-warning ch. indicating the end of the tape.

6. Summary.—The Ferut System has the following features:

(1) No reference to fixed magnetic tracks is made except in a special list called the directory.

(2) Minimum time and mental effort are consumed in changing from one routine to another during the operation of a problem.

(3) The details of the input are easy to remember.

(4) Alterations may be made to routines or pages of data with a minimum of effort.

(5) Certain automatic checks are built into the system.

(6) Library tapes may be used without alteration in a problem tape.

This system of input and routine organization was adopted after studying the various systems in use at the University of Manchester.[1] The details were devised by the author in cooperation with C. STRACHEY, National Research Development Corporation, London, England, and H. GELLMAN, Chalk River, Canada, with the advice and criticism of C. C. GOTLIEB and the staff of the Computation Centre, University of Toronto.

J. N. P. HUME

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[1] *The Programmers Handbook for the Manchester Electronic Computer (Mark II)*. mimeographed, Univ. of Manchester, England, Aug. 1952, 9 chapters.
[2] M. V. WILKES, D. J. WHEELER & S. GILL, *The Preparation of Programs for an Electronic Digital Computer*. Addison-Wesley Press, Inc., Cambridge, Mass., 1951.
D. R. HARTREE, *Numerical Analysis*. Chapter XII, Oxford University Press, 1952.
[3] Anon., "The incorporation of subroutines into a complete problem on the NBS Eastern Automatic Computer," *MTAC*, v. 4, 1950, p. 164–168.
[4] D. J. WHEELER, *Program Organization for the University of Illinois Digital Computer*. Internal Report No. 33, University of Illinois, May 16, 1952.

DISCUSSIONS

Programs for Computing the Hypergeometric Series

The hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

appears in the solution of various problems in applied mathematics and it is therefore desirable to have efficient methods for evaluating this series. To meet these needs a program of investigating available methods for computing the hypergeometric series is being conducted. The present note discusses two computations on SEAC.

A code for real values of parameters and argument has been checked out with a, b, c equal to various combinations of 1, 2, 3 and $-.9 \leqq x \leqq .9$ where many of the answers are known functions. The results were accurate to nine and more often ten significant figures. For each set of parameters it took approximately twenty-five minutes to tabulate the series for $x = -.9(.1).9$.

A code for complex values of parameters and argument has been checked with one of the real cases above giving the same accuracy, and with $a = 1 + i.2$, $b = -2$, $c = 2 + i$, $z = .5 - i.8$ where the hypergeometric series is a polynomial. The agreement in this case was exact.

A previous hand computation of $F(a, b; c; z)$ for

$$a = -11.753 + i12.204, \quad b = 12.753 - i12.204, \quad c = 2, \quad z = .5$$

gave the result

$$F(a, b; c; z) = 832{,}109 - i827{,}535$$

where the result was accurate to four figures. Using these same parameters and argument and carrying all computations to at least five decimal places and computing until the truncation error was less than 10^{-2} (54 terms) SEAC gave

$$F(a, b; c; z) = 832{,}103.8 - i827{,}536.8.$$

Repeating the computations to at least seven decimal places and computing first until the truncation error was less than 10^{-3} (57 terms) and then until the truncation error was less than 10^{-5} (65 terms) SEAC gave the identical results

$$F(a, b; c; z) = 832{,}108.0985 - i827{,}540.7310.$$

As a check on this result $F(a - 1, b; c; z)$ and $F(a, b + 1; c + 1; z)$ were computed and the following recursion formula yielded the results

$$F(a, b; c; z) = F(a - 1, b; c; z) + bzF(a, b + 1; c + 1; z)/c$$

$$F(a, b; c; z) = 827{,}563.5390 - i46{,}025.5154 + bz(123{,}187.2697 - i127{,}239.3492)/c$$

$$F(a, b; c; z) = 832{,}108.0972 - i827{,}540.7303$$

which agrees with the previous computation to the second decimal place. Each of the above computations took four minutes time on SEAC, including reading in and out.

This code uses double precision operations and begins by reading in a, b, c, z and the size of the term where the series is truncated, where all are twenty-two digit decimal numbers. After computing and printing the result the SEAC calls for a new z and will compute using the previous parameters, or may be made to read in new parameters by sending the Control to address zero. In general the result will be accurate to 10^{11-8} significant figures where $F(a, b; c; z)$ and its largest term are both less than 10^8.

This last code has been modified to compute the confluent hypergeometric function for complex parameters and argument. Using the modified code $w = 1.35218\,220994$ has been found as a zero of the function[1] $F\left(\dfrac{1 - w}{2}; 1; 2w\right)$.

W. F. Cahill

NBSCL

[1] H. A. Lauwerier, "The use of confluent hypergeometric functions in mathematical physics and the solution of the eigenvalue problem," *Applied Scientific Research*, 1951, A, z, p. 184–204.

Bibliography Z

1091. Anon., "Analog-to-Digital Converter simplifies data reduction," *Product Engineering*, v. 24, May 1953, p. 234–235.

Principles of the SADIC analog-to-digital converter built by Consolidated Engineering Corp. are briefly described.

R. D. Elbourn

NBSCML

1092. Anon., "Computer assembly line," *Electronics*, v. 26, July 1953, p. 200.

This is a short expository paragraph about the IBM electronic data-processer Model 701.

1093. Anon., "Converters put data into useful form," *Aviation Week*, v. 58, May 18, 1953, p. 68.

Analog-to-digital converters having three decimal digit precision and a cycle slightly less than one second have been announced by two companies, the SADIC by Consolidated Engineering Corp. and the Teleducer by Telecomputing Corp.

R. D. Elbourn

NBSCML

1094. Anon., "Mathematicians needed to help 'brain,'" *Machine Design*, May 1953, v. 25, p. 236.

Anon., "Armour Research Foundation to get electronic super-brain," *Midwest Engineer*, 5, Feb. 1953, p. 28.

Both articles deal with the IBM electronic computer 701. The first gives a brief description of this card-programmed calculator with its magnificent equipment of electrostatic memory, magnetic drum, and magnetic tapes. The second announces the prospective rental of this model by the ARF of the Illinois Institute of Technology.

The pleasure we find in reading of any advancement in the field of electronic computation is considerably dampened by the persistent occurrence in print of such epithets as "super-brain," "thinking machine," and the like—as applied to computers. Nor is it a pleasure for the hard-working mathematician to be told that "fantastically hard problems are easily solved by the electronic computers."

It should be realized that man-made machines cannot solve problems, any more than telephones can carry on conversations, or telescopes can make celestial observations. None but the *human* mind is capable of initiating and solving mathematical problems, as well as of utilizing various types of equipment for the computation of the results.

Our ruffled feathers were smoothed somewhat by the timely warning voiced in the first article that "there may not be enough top quality mathematicians to prepare problems for . . . these machines. . . ." Prophetic and true words!

I. Rhodes

NBS

1095. Anon., "NBS designs 'building block' computer," *Product Engineering*, v. 24, March 1953, p. 205.

A new computer, under development at the National Bureau of Standards for the Defense Department, is composed largely of approximately 800 standardized circuit packages. These are expected to take care of 90 percent of the total circuitry, exclusive of memory units. Two general types of standardized packages have been developed; the first contains an amplifier tube, pulse transformer and germanium diodes; and the second, delay lines. These basic types are made in several versions as dictated by circuit requirements.

The assemblies, now being manufactured commercially, appear to be about two inches wide, five inches high and 10 inches deep. Connection is through a 60-contact pin assembly at the rear. To facilitate checking and fault-localization, a test jack is located on the front of each unit.

M. L. Greenough

NBS Electronic Instrumentation

1096. An Wang, "High-speed number generator uses magnetic memory matrices," *Electronics*, v. 26, May 1953, p. 200–204.

Numbers are displayed in an 8×8 dot pattern on a cathode ray tube by using conventional raster-scan deflection and pulsing the beam on

according to character shapes stored in 8×8 magnetic core matrices. The device has been tested at 8,000 characters per second, but the builder believes 100,000 per second is achievable.

R. D. Elbourn

NBSCML

1097. W. Cochran & A. S. Douglas, "A new application of EDSAC to crystal structure analysis," *Nature*, v. 171, June 20, 1953, p. 1112–1113.

Determination of the structure of centro-symmetric crystals by evaluating the electron density for each of the 2^N possible sign combinations on N structure factors can be made after the correct signs of about twenty of the largest structure factors have been found (for crystals of moderate complexity).

The EDSAC surveys this large number of alternative answers and selects from them a few which fulfill certain criteria. In the case of salicylic acid, it completed in less than three hours the calculation for $2^{16} = 65{,}536$ independent possibilities. The number of sign combinations corresponding to each such possibility is listed in an accompanying table in this article. This method is considered valuable when used in conjunction with Pepinsky's X.R.A.C. for rapid testing of the most plausible sign combinations.

Mention is made of another method using EDSAC which requires only a few minutes of machine time for determining rapidly the structures of crystals of moderate complexity.

E. Marden

NBSCL

1098. P. Klass, "Giant brains could aid air defense," *Aviation Week*, v. 58, May 11, 1953, p. 67–68+.

This article is a description of two new computing machines, Remington Rand's 1103 and International Business Machines Corporation's 701; it stresses their possible uses in the aviation field. The two uses mentioned were both "real time" control uses, i.e., air defense operations and airport traffic control. No mention was made of their main use so far, that of solving complex problems and simulating flight conditions. "Real time" control can consist of either giving fast answers pictorially and numerically to human beings or of actually controlling aircraft and guided missiles by remote control. Aside from the vast amount of pre-stored information, information as to changing conditions can be fed into the computer directly from radar and other equipment, thus enabling the machine to make its lightning computations.

The bulk of the article contrasts the specifications for the two machines, giving a good summary of their technical points. The machines are roughly comparable but quite different in some details.

C. J. Swift

NBSCL

News

Symposium on Monte Carlo Methods.—A Symposium on this subject is being sponsored by the Aeronautical Research Laboratory, WADC. It will be held at the University of Florida at Gainesville, on March 16 and 17. Those interested in Monte Carlo methods

are invited to attend. Further information may be obtained from H. A. MEYER, University of Florida, Gainesville.

Association for Computing Machinery.—A general meeting of the Association was held at the Massachusetts Institute of Technology on September 9, 10, and 11, 1953. Many new ideas and developments in the applications of computing machinery were reported in parallel sessions. In addition the Whirlwind I Computer located at MIT was demonstrated for those attending. The program for the meeting was as follows:

September 9, 1953, Morning Session

Punched card techniques	F. M. VERZUH, MIT, *Chairman*
Solution of simultaneous linear equations by the Crout method on the IBM card programmed calculator	K. J. BERG, United Aircraft Corp., East Hartford, Conn.
Remington Rand 409 Electronic Computer as applied to scientific and commercial problems	R. T. BRUCE, Remington Rand, Inc., South Norwalk, Conn.
Matrix multiplication on standard punched-card machines	JAMES MORISON, Douglas Aircraft Co., Inc., Santa Monica, Calif.
A calculation system for the IBM CPC Model 1A	J. C. REA, Engineering Calculations Group, Allison Division, General Motors Corp., Indianapolis, Indiana
The EIP—An external and internal program setup for IBM's Model II CPC	J. C. SHAW, The Rand Corp., Santa Monica, Calif.
A three factor floating decimal panel for the IBM CPC	D. S. BYRON, Bell Aircraft Corp., Buffalo, N. Y.
A high-speed multi-purpose board for the IBM card-programmed electronic calculator	J. A. POSTLEY, Hughes Research and Development Laboratories, Culver City, Calif.
Planned "work flow" in an engineering computing installation	REX RICE, JR., Northrup Aircraft, Inc., Hawthorne, Calif.

September 9, 1953, Morning Session

Digital computer techniques	C. W. ADAMS, MIT, *Chairman*
Single vs. triple address computing machines	C. C. ELGOT, Naval Ordnance Laboratory, White Oak, Md.
A method of radix conversion	WALTER SODEN, Analysis Division, U. S. Naval Air Missile Test Center, Point Mugu, Calif.
Interim and comprehensive systems of computation on MIDAC	J. H. BROWN and J. W. CARR, Univ. of Mich., Willow Run Research Center, Ypsilanti, Mich.
Matrix multiplication on the ERA 1103	A. E. ROBERTS, Engineering Research Associates, Division of Remington Rand, Inc., Arlington, Virginia
Digital computer mathematics	J. W. MAUCHLY, Remington Rand, Inc., *Chairman*
A new method of determination of all roots of an algebraic equation	J. W. CARR, Univ. of Mich., Willow Run Research Center, Ypsilanti, Mich.
A comparison of machine methods for evaluating certain mathematical functions	W. B. FRITZ, Ballistic Research Laboratories, Aberdeen Proving Ground, Aberdeen, Md.
The numerical solution of semilinear parabolic equations by difference methods	A. J. PERLIS, Purdue Univ., Statistical Laboratory, Lafayette, Indiana

The continued-fraction algorithm for computing machines	G. W. King, International Telemeter Corp., Los Angeles, Calif., and L. J. Edson, Arthur D. Little, Inc., Cambridge, Mass.

September 9, 1953, Afternoon Session

Numerical analysis	F. B. Hildebrand, MIT, *Chairman*
A survey of methods of solving systems of linear algebraic equations	J. H. Curtiss, New York Univ., New York City
A survey of methods for dealing with eigenvalue problems	H. H. Goldstine, The Institute for Advanced Study, Princeton, N. J.
Problems on partial differential equations	F. J. Murray, Department of Mathematics, Columbia Univ., New York City

September 9, 1953, Afternoon Session

Recent systems developments	C. C. Hurd, IBM Corp., *Chairman*
MIDAC—The Michigan Digital Automatic Computer	J. E. DeTurk, R. Hock, J. Kaufman, & H. Bethel, Univ. of Michigan, Ann Arbor, Mich.
The IBM Magnetic Drum Calculator Type 650	F. E. Hamilton and E. C. Kubie, IBM Corp., Endicott, N. Y.
Storage class control in the ERA 1103	W. G. Welchman, Engineering Research Associates, Div. of Remington Rand, Inc., Arlington, Va.
The problems of preparing acceptance tests for digital computers	P. Brock & S. M. Rock, Consolidated Engineering Corp., Pasadena, Calif.
Automatic strain-gage and thermocouple recording on punched cards	R. Perley, United Aircraft Corp., East Hartford, Conn.
The analyzing reader	D. H. Shepard, Intelligent Machines Research Corp., Arlington, Va.
Computation for numerical control	J. H. Runyon, Servomechanisms Laboratory, MIT, Cambridge, Mass.
Large-scale computer output through the electronically controlled typewriter	L. W. Sprinkle, Mathematical Computation Branch, Air Force, Alexandria, Va.

September 10, 1953, Morning Session

Digital computer programming	A. S. Householder, Oak Ridge National Laboratory, *Chairman*
An interpretive routine for the manipulation of expansions in Boolean algebra	D. E. Muller, Digital Computer Laboratory, Univ. of Illinois, Urbana, Ill.
The IBM 701 speedcoding system	J. W. Backus, New York Scientific Computing Service, IBM Corp., N. Y.
The editing generator	A. M. Koss & J. H. Waite, Remington Rand, Inc., Philadelphia, Pa.
Analytical differentiation by a digital computer	H. G. Kahrimanian, Remington Rand, Inc., Philadelphia, Pa.
Digital computer programming	M. V. Wilkes, Cambridge, England, *Chairman*
Résumé of automatic coding techniques being developed for digital computers	J. M. Bennett, Ferranti Ltd., Manchester, England
	N. Rochester, IBM Corp., Poughkeepsie, N. Y.

	G. M. Hopper, Remington Rand, Inc., Philadelphia, Pa.
	C. W. Adams, Digital Computer Laboratory, MIT, Cambridge, Mass.
	S. Gill, University Mathematical Laboratory, Cambridge, England
	D. J. Wheeler, Univ. of Illinois, Urbana, Ill.
	J. W. Carr, III, Univ. of Michigan, Ypsilanti, Mich.

September 10, 1953, Morning Session

Punched card mathematics	C. C. Lin, MIT, *Chairman*
A method for solving boundary value problems of mathematical physics on punch card machines	S. Bergman, Applied Mathematics and Statistics Laboratory, Stanford Univ., Palo Alto, Calif.
A method of determining plate bending by use of a punched card machine	A. D. Wasel, Mathematics Dept., Univ. of Santa Clara, Santa Clara, Calif.
Computing supersonic flow around axial symmetric bodies using method of characteristics	M. Robinson, Bell Aircraft Corp., Buffalo, N. Y.
Control panels for a card programmed 604 computing system	N. A. Patton, Lewis Flight Propulsion Laboratory, Cleveland, Ohio
Logical algorithms	S. H. Caldwell, MIT, *Chairman*
Derivation of a prediction function for complex system performance	E. D. Fullenwider, Missile Evaluation Laboratory, NBS, Corona, Calif.
Extension of the Veitch chart method in computer design	R. W. Brooks, Raytheon Manufacturing Co., Waltham, Mass.
Automatic series-parallel circuit simplification	R. J. Nelson, IBM Corp., Endicott, N. Y.
Automatic analytic geometry in machine design	L. P. Crosman, Laboratory of Advanced Research, Remington Rand, Inc., Norwalk, Conn.

September 11, 1953, Morning Session

Numerical solution of partial differential equations	R. F. Clippinger, Raytheon Manufacturing Co., Waltham, Mass., *Chairman*
Finite-difference approximations to the fundamental frequency of a vibrating membrane	G. E. Forsythe and B. F. Handy, NBS, Los Angeles, Calif.
Boundary conditions in random walks	W. F. Bauer, Univ. of Michigan, Ypsilanti, Mich.
Numerical treatment of a fourth order parabolic partial differential equation	S. H. Crandall, Dept. of Mechanical Engineering, MIT, Cambridge, Mass.
Numerical solutions of some nonlinear heat-transfer problems	F. Alt, NBS, Washington, D. C.
Operation of a computation center	W. J. Eckert, IBM, *Chairman*
Operation of a technical computing facility	W. D. Bell, Telecomputing Corp., Burbank, Calif.
United Aircraft Corporation—Trinity College, Computation Course	W. Ramshaw, United Aircraft Corp., Machine Computation Laboratory, East Hartford, Conn.
The educational aspects of operating a computing service	G. T. Hunter and D. R. Mason, IBM Corp., N. Y.

Installation of an IBM 701	H. R. J. Grosch, General Electric Co., Evendale, Ohio

September 11, 1953, Morning Session

Analog computation	G. A. Philbrick, Philbrick Research Associates, *Chairman*
Automatic computation by analogue computers of certain errors in flight simulation problems	G. L. Landsman, NBS, Corona, Calif.
Solution of certain partial differential equations by analogue computers	O. L. Bowie, Watertown Arsenal, Watertown, Mass.
Analog computer solution of partial differential equations	H. B. Belck, Rensselaer Polytechnic Institute, Computer Laboratory, Troy, N. Y.
Analog computer approach to heat transfer problem	C. F. Kayan and V. Paschkis, Columbia Univ., N. Y.
A method of approximating the real roots of a polynomial using an analogue computer	R. E. Carroll, Bell Aircraft Corp., Buffalo, N. Y.
Analog computation	W. W. Seifert, MIT, *Chairman*
Equipment reliability as applied to analogue computers	H. Jacobs, Jr., MIT, Dynamic Analysis and Control Laboratory, Cambridge, Mass.
Generation of a function of two variables	R. P. Jerrard and G. T. Jacobi, General Electric Co., Schenectady, N. Y.
Generator for functions of two variables	G. R. Welti, MIT, Dynamic Analysis and Control Laboratory, Cambridge, Mass.
Survey of analogue multiplication	C. M. Edwards, Bendix Research Laboratory, Detroit, Mich.
An operational digital divider	M. A. Meyer, B. M. Gordon, and R. N. Nicola, Laboratory of Electronics, Inc., Boston, Mass.

September 11, 1953, Afternoon Session

Operation of a computation center	C. V. L. Smith, ONR
The University Computing Laboratory	A. J. Perlis, Purdue Univ., The University Computing Laboratory, Lafayette, Ind.
Operation of the Office of Statistical Services	F. M. Verzuh, MIT, Office of Statistical Services, Cambridge, Mass.
Operating a computer efficiently	C. C. Gotlieb, Computation Centre, Univ. of Toronto, Toronto, Canada
Operating and administrative procedures for MIDAC	D. Neeb, Univ. of Michigan, Ypsilanti, Mich.
Problems arising in the administration of a multi-machine digital computing service	J. W. Fischbach, Ballistic Research Laboratories, Aberdeen, Md.
An integrated data processing facility	E. M. McCormick and H. H. Rosen, Missile Evaluation Laboratory, National Bureau of Standards, Corona, Calif.

September 11, 1953, Afternoon Session

Business data handling	R. A. Mangini, John Hancock Life Insurance Co.
Premium billing performed by large-scale computers	R. T. Wiseman, Sun Life Assurance Co. of Canada, Montreal, Canada
Periodic billing and accounting	E. F. Cooley, Prudential Insurance Co. of America, Newark, N. J.
The development and application of electronic equipment in Monsanto's accounting department	E. J. Cunningham, Monsanto Chemical Co., St. Louis, Mo.
Experience with the Census UNIVAC	D. H. Heiser & J. L. McPherson, Bureau of the Census, Department of Commerce, Washington, D. C.
Budget computation on IBM 701	Lt. R. E. Utman, Navy Aviation Supply Office, Philadelphia, Pa.

OTHER AIDS TO COMPUTATION

Bibliography Z

1099. Anon., "Chart for the resolution of angles," *Product Engineering*, v. 22, no. 6, 1951, p. 187.

To be used for representing spatial figures on a plane.

1100. Anon., "High speed response in electromechanical integrator," *Product Engineering*, v. 24, no. 6, 1953, p. 240.

A combination of variable speed drive and d.c. feed back integrator constitutes an integrator whose input is a voltage regarded as a function of the time and with mechanical counter output.

1101. E. Bromberg & R. D. McCoy, "Calculating machines—new tools for the designer," *Product Engineering*, v. 22, no. 3, 1951, p. 85–88.

This article contains brief descriptions and photographs of REAC equipment and components including the servo multiplier and resolver and function generator. Also various applications to engineering problems are described.

1102. F. W. Bubb & W. L. Morris, "How analogical computing devices can serve process industries," *Chem. Engineering*, v. 57, no. 7, 1950, p. 142–144.

1103. W. H. Burrows, "Methods of calculating with graph papers," *Product Engineering*, v. 22, no. 4, 1951, p. 140–145; also no. 6, p. 168–171.

1104. F. P. Cozzone, "Organizing a computer center in the engineering department," *Product Engineering*, v. 23, no. 1, 1952, p. 136–141.

Various types of digital and analogue computers are described. A chart is given, showing the relative suitability of these types for aircraft design

problems. The organization, personnel problems, power requirements and space layout for an engineering computing center are considered.

F. J. M.

1105. A. S. Hall & D. C. Tao, "Analysis of a symmetrical five-bar linkage," *Product Engineering*, v. 23, no. 1, 1952, p. 175–177; also "Design charts for a five-bar linkage," p. 201, 203, 205.

1106. B. H. List, R. C. McMaster, & R. L. Merrill, "Analogous systems in engineering design," *Product Engineering*, v. 24, no. 1, 1953, p. 184–195.

Various network type analogues and a differential analyzer are discussed.

1107. C. P. Nachod, "Nomograph for the volume of cones or pyramids," *Product Engineering*, v. 22, no. 5, 1951, p. 209.

1108. D. W. Peaceman & J. E. Vivian, "Bantam differential analyzer," *Chem. Engineering*, v. 57, no. 8, 1950, p. 106–107.

This article describes a small differential analyzer built by A. B. Macnee and using his multiplying unit.

1109. D. H. Pickens, "The electronic analog computer," *Product Engineering*, v. 24, no. 5, 1953, p. 176–185.

Principles and certain engineering applications of an electronic differential analyzer based on a feed back d.c. amplifier are discussed.

1110. W. W. Soroka, "Equivalent dynamical systems for complex vibration problems," *Product Engineering*, v. 23, no. 7, 1952, p. 130–133.

A method of setting up a mechanical analogue with lumped masses and linear springs for structural vibration problems is described.

1111. W. W. Soroka, "Resistance network analogue for solving vibration problems," *Product Engineering*, v. 22, no. 4, 1951, p. 103–105.

A manually adjusted resistance network is described.

1112. E. C. Varnum, "Circular nomogram theory and construction technique," *Product Engineering*, v. 22, no. 8, 1951, p. 152–156.

An explanatory article with engineering examples.

1113. E. C. Varnum, "Nomogram for evaluating test data," *Product Engineering*, v. 24, no. 2, 1953, p. 215.

A circular nomogram for applying the "*t* test" to determine the significance of a difference of means.

NOTES

157.—Analytical Approximations. [See also Note **153.**] The first twelve approximations concern the functions $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ in which $I_0(x)$ and $I_1(x)$ are the usual Bessel functions of imaginary argument.

(35) To better than .0006 over $(0, \infty)$,

$$e^{-x}I_0(x) \doteq \left(\frac{1 + .297x + .341x^2}{1 + 2.333x + 2.137x^2 + 2.096x^3}\right)^{\frac{1}{2}}.$$

(36) To better than .00005 over $(0, \infty)$,

$$e^{-x}I_0(x) \doteq \left(\frac{1 + .302x + .234x^2 + .114x^3}{1 + 2.2979x + 2.3871x^2 + 1.2032x^3 + .7183x^4}\right)^{\frac{1}{2}}.$$

(37) To better than .000,009 over $(0, 1)$,

$$e^{-x}I_0(x) \doteq (1 + .0302x + .0889x^2)/(1 + 1.0299x + .3728x^2).$$

(38) To better than .0001 over $(0, 2)$,

$$e^{-x}I_0(x) \doteq (1 + .1693x + .0844x^2)/(1 + 1.1665x + .5247x^2).$$

(39) To better than .0007 over $(0, 4)$,

$$e^{-x}I_0(x) \doteq (1 + .4537x + .0955x^2)/(1 + 1.4387x + .8855x^2).$$

(40) To better than .0005 over $(0, \infty)$,

$$e^{-x}I_1(x) \doteq x(3.78 + 9.81x + 3.09x^2 + 6.36x^3)^{-\frac{1}{2}}.$$

(41) To better than .0003 over $(1, \infty)$,

$$e^{-x}I_1(x) \doteq x(4.51 + 9.00x + 3.25x^2 + 6.37x^3)^{-\frac{1}{2}}.$$

(42) To better than .00005 over $(2, \infty)$,

$$e^{-x}I_1(x) \doteq x(10.281 + 3.752x + 4.541x^2 + 6.296x^3)^{-\frac{1}{2}}.$$

(43) To better than .000,006 over $(0, 1)$,

$$e^{-x}I_1(x) \doteq (.49974x - .01695x^2)/(1 + .95935x + .36282x^2).$$

(44) To better than .00007 over $(0, 2)$,

$$e^{-x}I_1(x) \doteq (.4981x + .0066x^2)/(1 + .9805x + .4477x^2).$$

(45) To better than .0003 over $(0, 4)$,

$$e^{-x}I_1(x) \doteq (.4935x + .0268x^2)/(1 + .9667x + .5373x^2).$$

(46) To better than .00008 over $0 \leq x \leq \infty$,

$$e^{-x}I_1(x) \doteq x(15.4 + 74.8x + 67.2x^2 + 235.8x^3 + 43.5x^4 + 59.4x^5 + 39.6x^6)^{-\frac{1}{4}}.$$

(47) Definite Integral: To better than .00055 over $0 \leq x \leq \infty$,

$$N(x) = 30\pi^{-4}\int_0^\infty t^7(e^{t^2} - 1)^{-1}e^{-(x/t)^7}\,dt$$

$$\doteq (1 + .38382x^6 - .55605x^7 + .34791x^8 - .10369x^9 + .01245x^{10})^{-1}.$$

(48) Mach Number in Terms of Pressure Ratio: To better than .0011 over $.3 \leq M \leq 1.0$, the inverse of (a) is (b).

(a) $x = P_S/P_R = [1 + (\gamma - 1)M^2/2]^{-\gamma/(\gamma-1)}, \quad \gamma = 1.4.$

(b) $M \doteq (2.714 - 2.625x)/(1 + 1.650x - 1.955x^2).$

(49) Mach Number in Terms of Pressure Ratio: To better than .0014 over $1 \leq M \leq 3$, the inverse of (c) is (d).

(c) $$x = P_S/P_R = [2\gamma M^2/(\gamma + 1) - (\gamma - 1)/(\gamma + 1)]^{1/(\gamma-1)} \times [(\gamma + 1)M^2/2]^{-\gamma/(\gamma-1)}, \quad \gamma = 1.4.$$

(d) $M \doteq (8.19 + 29.40x - 24.58x^2)/(1 + 30x).$

(50) Mach Number in Terms of Pressure Ratio: To better than .0021 over $.3 \leq M \leq 3$, the inverse of (a) over $.3 \leq M \leq 1$ conjoined with (c) over $1 \leq M \leq 3$ is given by

$$M \doteq (8.11 + 23.60x - 39.66x^2 + 8.98x^3)/(1 + 28.70x - 15.99x^2 - 5.74x^3).$$

Essentially the same approximation was reported to us by Mr. Philip Rapp.

(51) Pearson Cosine Transformation: To better than .0014 over $0 \leq x \leq 1$,

$$r(x) = \cos \pi/(1 + x^{1/2}) \doteq (-1 - 7.47x + 8.47x^2)/(1 + 11.65x + 12.05x^2).$$

$r(x^{-1}) = -r(x)$ can be used to obtain function values over $1 \leq x \leq \infty$.

(52) Pearson Cosine Transformation: To better than .00017 over $0 \leq x \leq 1$,

$$r(x) \doteq \frac{-1 - 4.828\eta + 7.866\eta^2 - 2.038\eta^3}{1 + 5.560\eta - 4.985\eta^2 + .385\eta^3}, \quad \eta = \frac{x}{.16 + .84x}.$$

(53) Natural Addition Logarithm: To better than .00026 over $0 \leq x \leq \infty$, $\ln(1 + e^{-x}) \doteq (1 + .3581x + .1151x^2 + .0094x^3 + .0052x^4)^{-2} \ln 2.$

(54) Natural Addition Logarithm: To better than .000,045 over $0 \leq x \leq \infty$,

$$\ln(1 + e^{-x}) \doteq (1 + .36123x + .10204x^2 + .02411x^3 - .00055x^4 + .00069x^5)^{-2} \ln 2.$$

(55) Natural Addition Logarithm: To better than .000,008 over $0 \leq x \leq \infty$, use constants .360571, .105546, .018760, .002654, −.000100, .000066 in next form of sequence.

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158.—Counts of Twin Primes less than 100000. The writer has constructed a table making the counting of twin primes relatively easy. The counts differ from those of previous observers to such an extent that it seems desirable to publish the results.

The previous tables are identified merely by giving the author's names. The exact references are given in LEHMER's *Guide*.[1]

Table I shows the various counts of twin primes for each of the 10 myriads. Following GLAISHER 1 and 3 are counted as pair of twin primes.

TABLE I

Myriad	SUTTON	STÄCKEL	POLETTI	GLAISHER & SEXTON
1	206	206	203	206
2	137	137	135	137
3	125	125	125	125
4	124	124	124	124
5	113	114	113	114
6	106	106	105	106
7	93	94	94	94
8	102	102	102	102
9	108	109	110	109
10	108	108	105	108
Totals	1222	1225	1216	1225

The results of GLAISHER agree exactly with those of the author. POLETTI gives a total of 1217 but the figures he gives add in fact to 1216.

The total 1225 is in agreement with HARDY & LITTLEWOOD.

Table II shows the distribution of these twin primes as to their final digits.

TABLE II

	Digital Endings			
Myriad	(1, 3)	(7, 9)	(9, 1)	*Total*
1	68	64	72	204
2	49	46	42	137
3	42	45	38	125
4	46	39	39	124
5	36	40	38	114
6	39	32	35	106
7	24	36	34	94
8	34	33	35	102
9	33	35	41	109
10	31	44	33	108
Totals	402	414	407	1223
POLETTI	401	411	403	1215

The reference here is to POLETTI 2, Table XVII.

The author has made a count of double prime pairs like 191, 193, 197, 199. The number n_r of these in the r-th myriad is as follows:

r:	1	2	3	4	5	6	7	8	9	10
n_r:	11	7	3	2	1	2	3	3	3	2

There are in all 37 such double pairs. The set 1, 3, 7, 9 is not counted.

Primes of the forms $6n - 1$ and $6n + 1$ have also been counted and are compared with Poletti in Table III.

TABLE III

	$6n - 1$		$6n + 1$	
Myriad	POLETTI	SEXTON	POLETTI	SEXTON
1	616	616	614	611
2	519	520	513	513
3	498	497	486	486
4	479	479	479	479
5	463	463	467	467
6	466	466	458	458
7	449	449	429	429
8	447	447	455	455
9	437	436	442	440
10	432	433	444	446
Totals	4806	4806	4787	4784

The total 4784 agrees with the count given in CUNNINGHAM 28.

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[1] D. H. LEHMER, *Guide to Tables in the Theory of Numbers.* Nat. Res. Council, *Bull.* no. 105. Washington, 1941.

159.—A NEW APPROXIMATION TO THE RECIPROCAL OF π. In 1949 I decided to extend to about 1120D my previously published[1] approximation to π^{-1}. To attain a completely reliable result it was decided to evaluate the reciprocal of π by two methods: first, the summation of a suitable highly convergent series; and second, machine division of unity by an approximation to π that LEVI B. SMITH and I had computed to about 1120D[2] in extension of an 808D value we had previously published[3] in collaboration with D. F. FERGUSON.

Smith computed by desk machine the sum of the following series due to RAMANUJAN:[4]

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\,\frac{1}{2}\,\frac{1\cdot 3}{4^2} + \frac{44043}{882^5}\,\frac{1\cdot 3}{2\cdot 4}\,\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots$$

Each term, omitting the coefficients 1123, 22583, $\cdots$, in arithmetic progression, was evaluated to 1130D and then these coefficients were introduced in the calculation to complete the summation. Two independent congruential checks were applied at intervals of 100D at every stage of this calculation.

Concurrently I used the ENIAC approximation[5] to π as the divisor in an independent evaluation of π^{-1}. Smith's final result was compared with mine in October 1952 and complete agreement through 1123D was observed, thereby verifying to a comparable degree of accuracy the ENIAC determination of π.

Subsequently I continued the desk machine reciprocation of π to 1408D. As an exercise on the UNIVAC the complete product of this approximation to π^{-1} and the ENIAC 2040D work sheet value of π was computed on 20 April 1953 by two independent routines in an average time of about 8 minutes. The accuracy of my desk machine calculation was confirmed and the extension of π^{-1} to 2037D was at once deduced on the UNIVAC. The final result of these investigations is reproduced below.

$\pi^{-1} =$	0.31830	98861	83790	67153	77675	26745	02872	40689	19291	48091
	28974	95334	68811	77935	95268	45307	01802	27605	53250	61719
	12145	68545	35159	16073	78582	36922	29157	30575	59348	21463
	39967	84584	79933	87481	81551	46155	49279	38506	15377	43478
	57924	34795	32338	67247	80483	44725	80236	64760	22844	53995
	11431	88092	37801	73805	34791	22409	78821	87387	56881	71057
	44619	98928	86800	49734	46954	78919	22179	66461	93566	14981
	23339	72925	60939	88973	04375	76314	95731	33928	48207	79917
	48278	69721	99677	36198	39992	48857	51170	34235	77168	62235
	03753	43210	93095	07397	60194	78920	72951	86675	36118	60498
	89932	70610	65431	35510	06440	64955	56327	94332	04589	34962
	39196	33168	12120	33606	07199	62678	23974	99766	55733	08870
	55951	01400	32481	35512	87776	99142	62176	02443	98752	29536
	27555	29475	78126	61360	92915	95696	35226	24854	62813	99215
	50049	00059	55197	14178	11380	55935	70263	05042	00326	35492
	04184	96232	12481	12291	24062	92968	17849	69183	82870	42315
	08151	12401	74305	32136	04434	31828	15149	49165	44519	54925
	70799	75031	06587	81627	96354	48187	16509	59414	66574	38081
	39995	18153	15415	69869	40787	17965	61743	46851	28073	37902
	33250	91411	88665	52625	37300	05224	54359	42306	42251	99008
	77335	89007	52511	21672	63423	39051	95162	56449	88324	66686
	29021	22470	73757	12622	72733	84334	28413	94939	20258	50115
	66721	06239	21718	90196	79113	43741	99094	93020	86324	76310
	35161	67888	59599	41999	01050	87751	32258	89176	66136	92101
	57058	30302	82080	97859	77012	77632	15523	93986	14682	07799
	91573	83781	19618	74755	44123	75086	44543	78602	73251	05224
	77560	77507	77622	13628	13530	86816	56557	05386	68535	99112
	14158	07721	20705	47799	24902	51991	49855	25940	47188	19116
	86023	29659	28237	11554	24811	50889	89140	43579	53958	48189
	80654	58954	04332	99207	13063	63070	88007	68137	97494	35383
	17752	63819	33013	92880	95539	41375	36731	35562	09559	59090
	07067	91516	60376	36773	75875	53224	96299	06119	93116	04381
	67197	50207	02542	58086	46316	09974	39373	75551	89313	26924
	42068	40888	17109	95700	75854	77388	58707	32387	55658	57471
	87568	69406	46047	42916	75847	11423	72726	83858	92036	63645
	83928	33001	75661	58662	70699	55819	94917	29858	05349	01219
	78737	81891	76610	06740	61076	10946	24643	16188	63953	52064
	56626	28379	61949	96448	76670	34871	39796	95002	07900	13677
	60079	57344	71992	16048	00547	80217	49909	70957	58471	36522
	27989	78065	37994	85416	69922	29841	65780	75535	69486	07100
	91369	12167	34295	86169	13446	65407	09707	85		

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QUERY

43. Incomplete Hankel Function. Has the Incomplete Hankel function

$$f(x, y) = \int_x^\infty dt \exp(-iyt)/(t^2 - 1)^{\frac{1}{2}}$$

been tabulated? If so, where can such a table be obtained?

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CORRIGENDUM

V. 7, p. 275, l. —2, *for* Mag *read* Mac.